

# SUBJECT MODULARIZATION AND RESEARCH PROJECTS WITH HIGH SCHOOL STUDENTS ON MATHEMATICAL IMAGE PROCESSING

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## ABSTRACT

This article is concerned with a new mentoring strategy, called *subject modularization* (SM), for high school students to be able to make a fundamental contribution to research in mathematical image processing. In SM, both mathematical subjects and software are partitioned into small modules, each of which is simple and easy enough to be manageable for talented high school students. Then, the students can finish a research project successfully by resolving a series of easy-to-solve problems. Also we consider an effective strategy for mentors to be able to provide a balance of support so that the students can have a reasonable share of the work and make significant contributions to interdisciplinary research in applied mathematics. Through the project, two high school students have developed new mathematical models and corresponding numerical schemes for efficient and reliable image processing in denoising and segmentation, which are publishable to an academic journal. Various examples carried out by the students are shown to demonstrate effectiveness of the newly developed models and numerical schemes.

**Key words.** Subject modularization, image denoising, zooming, segmentation, anisotropic diffusion, mathematical image processing.

## 1. Introduction

It is often difficult for high school students (or undergraduates) to make a research collaboration with mathematicians on interdisciplinary research subjects publishable to an academic journal. It is mainly because research activities in applied mathematics require a reasonably deep understanding over mathematics, physics, computer sciences, and engineering, which has been believed to be beyond the scope of knowledge obtainable by undergraduate students. Thus a challenging task in education is to develop strategies for the students to be able to follow the guidance, share the work in research collaboration, and make important contributions.

In this article, we present an effective mentoring strategy and its outcome, in the area of mathematical image denoising and segmentation, implemented and carried out by two high school students. Image denoising is one of oldest problems in image processing; it is important in various image restoration tasks (medical imaging and remote sensing) and

is often necessary as a pre-processing for other tasks such as segmentation and registration. However, most denoising algorithms introduce nonphysical dissipation which makes the resulting images look blurry. We develop mathematical and numerical techniques to minimize such a drawback in denoising. For image segmentation, the major objective is to identify an image as a collection of parts, each of which has a strong correlation with real-world objects. Parts in an image can be separated by a contour. Thus, the practical goal of image segmentation is to divide the image into parts by inserting contours. We will consider a new level set segmentation algorithm which hybridizes the gradient-based method and the gradient-free method, for an efficient and reliable segmentation.

On the other hand, our educational objective is to construct a suitable environment for the students to be able to make fundamental contributions to such an interdisciplinary research in applied mathematics. We are aware of learning theories such as *information processing* [13], [19] and *social constructivism* [2], [11], [20]. We believe that learning occurs most effectively through information processing and social interaction between a more experienced and competent individual (i.e., a mentor) and a less competent individual (i.e., a student). Such interaction is the conduit through which knowledge is constructed, to help the learner internalize specific mathematical concepts.

The proposed research projects, presented in detail in Appendices, require both strategic development and large hours of computational experiments. Thus the problem appears too difficult and complicated for high school students to be involved in the research. However, we introduce a new strategy of arranging solution procedures for the students to achieve a great success, called *subject modularization* (SM). We first partition appropriately both mathematical subjects and software into small *modules*, each of which is manageable for the students. Then we provide the students with a *modelcode* to promote their success with computer implementation. A modelcode is a computer code, often incomplete, on which one can add or modify modules for various applications of interests. Composing of an effective modelcode along with an appropriate SM is crucially important for the mentors to provide a balance of support so that the students can have a reasonable share of the work and derive maximum pleasure from the research experience.

With an appropriate support from mentors, the students can successfully accomplish a research project by resolving and implementing a series of small easy-to-solve modules.

Regardless of our awareness of learning theory, the mentors have recognized in the early state of the project that it is necessary to develop strategies for effective teaching and research for the students. We have employed Bloom's taxonomy [1] as our guide in working with the talented students. For a given subject module, the upper three levels (analysis, synthesis, and evaluation) are more emphasized during the students' research work. However, we have not hesitated to address the problem in a lower level when the students show difficulties. In particular, during the weekly meeting, the students are encouraged to utilize "Polya's Four Steps" [17] in resolving the assigned task.

In the beginning stage of the project, the students show difficulties in understanding mathematical backgrounds and physical implications of the computational algorithm. However, after being experienced in computer implementation, they could understand what they were doing more thoroughly. Furthermore, they have been able to introduce new numerical schemes which are consistent and showing satisfactory properties in mathematical image processing.

A successful mentoring process requires the following:

- *Subject modularization* is a necessary step, because a research project (of which the results are publishable to academic journals) needs to incorporate various interdisciplinary concerns that are far beyond the knowledge level of the students.
- *A balanced support* is another important component for the students to develop independence and experience the joy of discovery [17]. The students are provided with a modelcode along with an appropriate SM.
- *Perceptible subjects first*: Perceptible subjects must be given first-hand to the students, rather than abstract subjects. It has been observed that after certain experiences on computer implementation, the students can easily capture the abstract concepts.

We are interested in curriculum development for computational mathematics in a college level. Based on the knowledge obtained from the mentoring process, a project is being carried out for college students to be able to experience not only textbook examples in computational methods but also research for industrial problems. Mathematics is increasingly recognized as the liberating and transcendent knowledge as it really is. We maintain that mathematics, as powerful an intellectual tool as writing and reading [14], can help students advance in their fields, both short-term in coursework and long-term in their professional career.

The article is organized as follows. The next section contains strategies for SM. In Section 3, we present numerical results implemented and carried out by two talented high school students. The students first implemented a computational algorithm for denoising and modified it for image segmentation. These algorithms have been numerically verified to be efficient and reliable for the removal of various noises (including impulse noise, Gaussian noise, and the checkerboard effect) and segmentation. Section 4 summarizes the mentoring process. Appendix A includes a mathematical formulation for image denoising: prelimi-

naries, a new mathematical model, and its corresponding anisotropic numerical schemes.

## 2. Subject Modularization

In this section, we present details of SM focusing on the denoising algorithm presented in Appendix A. Similar concerns can be applied to image segmentation.

The time-stepping procedure and anisotropic numerical schemes in Appendix A can be implemented in computers by high school students. In order to make a great success in the implementation, we provide them with a modelcode written in C++ and C. The original modelcode is composed to include modules such as the main function, input/output (IO) functions, and a driver function:

- **The main function**: Written in C++, it allocates the necessary memory dynamically, depending on the size of data, and calls input routines and a driver function.
- **IO functions**: Written in C, they read user-defined parameters and image files and write the results into files.
- **A driver function**: It is a protocol for the implementation of the computational algorithm in (A.5). It allows the students to add desired modules.

We also have provided the students with automatic generators (`cstart` and `mkmk`) [8]. Whenever the structure of the input data and/or user-specified parameters change, the students can run `cstart` to regenerate the main function and the input routines. Students can call the driver function by setting an inclusion file (to the main function); the routines generated by `cstart` do not need to be modified. The command `mkmk` produces a machine-dependent makefile, after analyzing the computer being utilized and the source files. These generators have saved a lot of time and made the project easier to achieve for the students and the mentors as well.

We have guided the students in such a way that different operators are implemented and saved into separate files as C-modules. In the following, the reader will see (a) how the denoising algorithm in Appendix A is partitioned for the students to implement successfully and (b) how the students can understand the algorithm *thoroughly*.

### 2.1. Computer implementation for the diffusion matrix

The major component for the computer implementation for the diffusion matrix  $\mathcal{A}$  presented in Appendix A is to get a function for the operator  $\mathcal{D}_x$  in (A.6). Note that the matrix  $\mathcal{A}_1$  in (A.9) can be easily organized from (A.7) and (A.10). The students are assigned to implement for  $\mathcal{D}_x$  and its counterpart for  $\mathcal{A}_2$ ,  $\mathcal{D}_y$ , only. Since the organization of matrices  $\mathcal{A}_1$  and  $\mathcal{A}_2$  (the operations in (A.7), (A.9), and (A.10)) is given in the modelcode as a function, the students can focus on the implementation of  $\mathcal{D}_x$  and  $\mathcal{D}_y$ . They have implemented the two desired functions successfully, with a certain help from the mentors for the treatment of formulas at boundary pixels.

## 2.2. Computer implementation for $\theta$ -ADI

In a for-loop, the  $\theta$ -ADI (A.5) has been partitioned into six parts as follows:

- (a) Get  $\mathcal{B}_\ell^{n-1}$ :  
 $\mathcal{B}[\ell] \leftarrow \mathcal{B}_\ell^{n-1}$ ,  $\ell = 1, 2$ ;
- (b) Compute and save  $\mathcal{B}_\ell^{n-1}u^{n-1}$ :  
 $\text{WS}[\ell] \leftarrow \mathcal{B}[\ell]\text{U}$ ,  $\ell = 1, 2$ ;
- (c) Get  $(1 + \theta\Delta t\mathcal{B}_\ell^{n-1})$ :  
 $\mathcal{B}[\ell] \leftarrow (1 + \theta\Delta t\mathcal{B}[\ell])$ ,  $\ell = 1, 2$ ;
- (d) Do the  $x$ -sweep:  
 $\text{WS}[1] \leftarrow (\text{U} - \Delta t[(1 - \theta)\text{WS}[1] + \text{WS}[2]])$ ;  
 $\text{WS}[1] \leftarrow (\mathcal{B}[1])^{-1}\text{WS}[1]$ ;
- (e) Do the  $y$ -sweep:  
 $\text{WS}[2] \leftarrow (\text{WS}[1] + \theta\Delta t\text{WS}[2])$ ;  
 $\text{WS}[2] \leftarrow (\mathcal{B}[2])^{-1}\text{WS}[2]$ ;
- (f) Swap the array for  $u^n$ :  
 $\text{U} \leftarrow \text{WS}[2]$ ;

where  $\text{U}$  is the array for the solution and  $\text{WS}[\ell]$  are arrays for temporary saving and the intermediate solution. Before swapping  $\text{WS}[2]$  for  $u^n$  into the array  $\text{U}$  in (2.1.f), one can measure the difference between  $\text{WS}[2]$  and  $u^{n-1}$  (saved in  $\text{U}$ ). The measured difference can be utilized as a stopping criterion for the diffusion iteration.

The above implementation involves two main operations: matrix-vector multiplication and matrix inversion (for tri-diagonal matrices). No specific difficulties have been observed from the students working on these modules one-by-one.

## 2.3. Derivation of numerical schemes

The major mathematical derivation of numerical schemes for denoising is related to the operator  $\mathcal{D}_x$  in (A.6), which is an approximation of  $\|\nabla \cdot\|$  at the mid of grid points in the  $x$ -direction. In an early stage of the project, students are assigned to derive alternatives of  $\mathcal{D}_x$ . However, such a mathematical research turns out to be far beyond the knowledge level of high school students. Thus we must have postponed the assignment and thus we began guiding them to focus on implementation of required modules, one at a time, as presented in the previous subsection. When the students finished major parts of implementation tasks, they are assigned *again* to derive alternatives of  $\mathcal{D}_x$ . Now, they could introduce well-designed numerical schemes which performed reasonably well in image denoising. One example is:

$$\mathcal{D}_x u_{i-1/2,j}^{n-1} = \frac{1}{2}(\mathcal{D}_c u_{i-1,j}^{n-1} + \mathcal{D}_c u_{i,j}^{n-1}), \quad (2.2)$$

where  $\mathcal{D}_c$  is the central second-order approximation of  $\|\nabla \cdot\|$  defined as

$$\mathcal{D}_c u_{i,j}^{n-1} = \left( (u_{i+1,j}^{n-1} - u_{i-1,j}^{n-1})^2 / 4 + (u_{i,j+1}^{n-1} - u_{i,j-1}^{n-1})^2 / 4 \right)^{1/2}.$$

The performance of (2.2) is quite similar to that of (A.6), except that it introduces slightly more numerical dissipation for some images.

Such a mathematical assignment can be considered as an evaluation method which allows the mentors to referee the students' level of understanding for the algorithm. From

the mentoring process for a mathematical image processing project of this kind, we have observed the following:

- Computer implementation is a necessary and first step for the students to achieve tangible results and provides a suitable atmosphere for mathematical motivation.
- Computer implementation can be appropriately modularized for the high school students to finish a research project in mathematics publishable to an academic journal.

The results will appear elsewhere as a research article [10]. The results have also motivated a mathematical analysis and applications to color image zooming [3].

## 3. Numerical Experiments

### 3.1. Denoising

As a synthetic noise, we consider random-valued additive Gaussian noise, which is scaled to be mean-zero and have a certain variance. The iterative algorithm (A.5) is stopped when the iterates satisfy

$$\frac{\max_{i,j} |u_{i,j}^n - u_{i,j}^{n-1}|}{\max_{i,j} |u_{i,j}^n|} \leq 0.01. \quad (3.1)$$

For all experiments, we set

$$\theta = \frac{1}{2}, \quad \Delta t = \sqrt{2}, \quad q = 1.7, \quad \varepsilon = 0.05.$$

The above parameters have been obtained heuristically from various numerical experiments and turn out to be very effective for image denoising for diverse real images. For such a numerical modeling, a deep understanding of the numerical schemes presented in Appendix A seems very crucial for the students to be able to explore effective parameters.

Figure 1 contains gray-scale Lenna images in  $256 \times 256$  cells. The noisy image is obtained by perturbing the original Lenna image by Gaussian noise of variance 488 (SNR=36). In seven iterations, the  $\theta$ -ADI (A.5) has converged satisfying the stopping criterion (3.1), with the recovered image depicted as in the bottom of Figure 1. As one can see from the figure, the noise has been effectively removed and image details are clearly preserved in the restored image.

Figure 2 presents another example for the performance of our new algorithm applied to image zooming. From the Lenna image, the face in  $50 \times 50$  cells is selected and zoomed by the  $C^2$  cubic interpolation by a factor of  $(8 \times 8)$  as in Figure 2(top). The zoomed image showed the so-called *checkerboard effect*. Figure 2(bottom) contains an enhanced image by the anisotropic diffusion. For this example, we choose  $\beta$  specifically designed for the elimination of checkerboard effects. As one can see from the figure, most of the artifact has been removed in the enhanced image in three iterations of the  $\theta$ -ADI.

### 3.2. Segmentation

In this subsection, we test effectiveness of a new hybrid segmentation model that has been introduced as a combination of a gradient-based model by Zhao *et al.* (ZCMO) [22] and a gradient-free model by Mumford-Shah-Chan-Vese (MSCV) [4], [15].



Fig. 1. Gray-scale Lenna image in  $256 \times 256$  cells: (top) the noisy image and (bottom) the restored image.

In Figure 3, we compare the performances of the MSCV model and our new hybrid model. The given image is a cross section of a human body around chest, downloaded from the site of the Visible Human Project. The MSCV model shows difficulties in many spots in the image; in particular, it could not locate edges correctly for the right upper part of the body. The drawback has been overcome with the new model. As one can see from the figure, our new model has detected the edges quite satisfactorily, in three ADI iterations. The new model incorporating the suggested numerical techniques is efficient and reliable for image segmentation.

#### 4. Conclusions

We have introduced the so-called *subject modularization* as a new mentoring strategy for high school students to be able to make fundamental contributions to research in applied mathematics. To facilitate students' success, mentors have first partitioned subject matter into small subject modules, each of which is manageable for talented high school students. Then we have provided the students with a *modelcode* to promote success in computer implementation. Two high school students have carried out projects in mathematical image denoising and zooming satisfactorily by solving and implementing a series of modules. The mentoring process has resulted in great successes both for

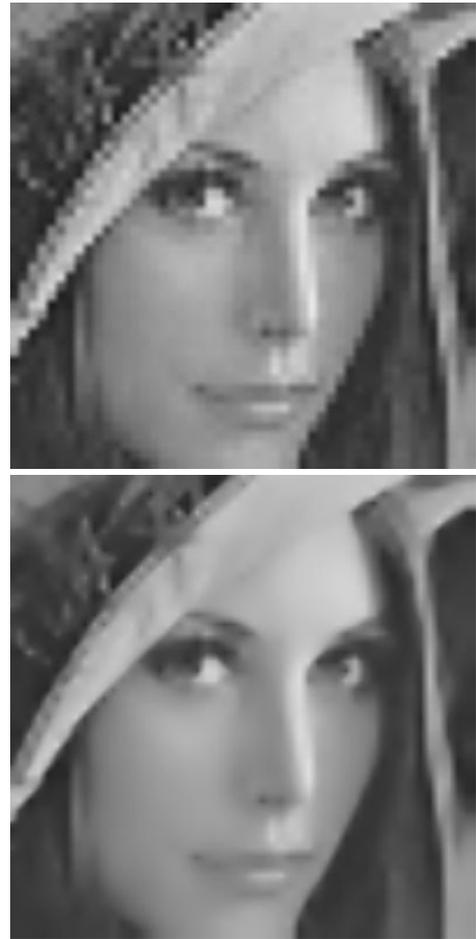


Fig. 2. Lenna Face in  $400 \times 400$  cells: (top) The enlarged image by the cubic interpolation from an image in  $50 \times 50$  cells and (bottom) an enhanced image with three iterations of the  $\theta$ -ADI.

education (by introducing innovative mentoring strategy) and for research (by developing new efficient and reliable models and their corresponding numerical schemes). The results will be published to an academic journal [10].

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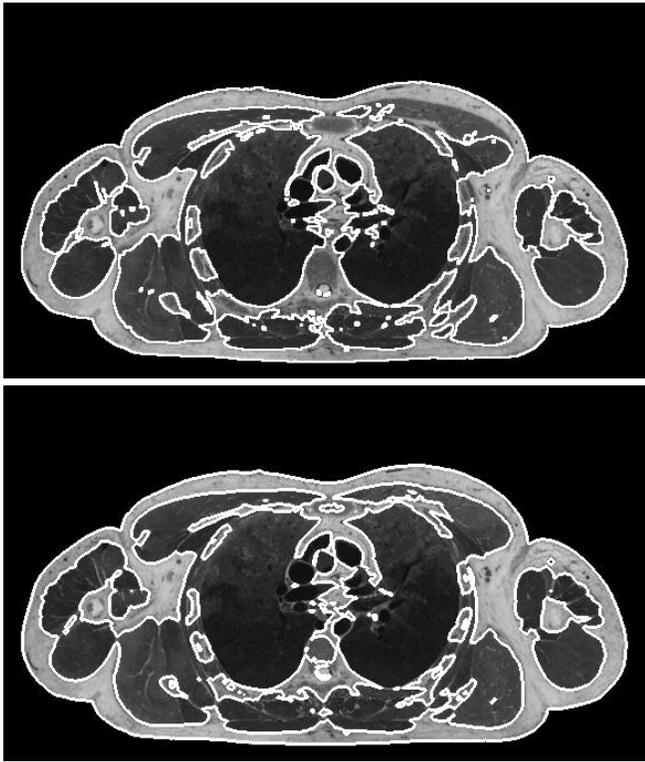


Fig. 3. Visible Human data: (top) the segmentation utilizing the MSCV model, and (bottom) the segmentation with the new model.

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## Appendix A. Denoising Methods

This appendix begins with preliminaries on mathematical denoising; we will introduce a new model, a nonlinear partial differential equation (PDE), for an efficient and reliable denoising. Then, we will discuss numerical schemes for the PDE.

### A.1. Preliminaries on mathematical denoising

Let  $f$  be the observed (noisy) image

$$f = u + e,$$

where  $u = u(\mathbf{x})$  is the image we want to find and  $e$  denotes the noise which can be Gaussian noise, impulse noise, or their combinations.

A common denoising technique is to minimize a functional of gradient, given as

$$\min_u \mathcal{F}_p(u), \quad \mathcal{F}_p(u) = \int_{\Omega} |\nabla u|^p d\mathbf{x} + \lambda \|f - u\|^2, \quad (\text{A.1})$$

where  $\lambda \geq 0$  and  $\Omega$  is the image domain. When  $p = 1$ , the first term in  $\mathcal{F}_1(u)$  is called the *total variation* (TV) [18]. Note that the restored image  $u$  becomes closer to  $f$  as  $\lambda$  grows. For  $0 < p < 1$ , the functional  $\mathcal{F}_p$  is non-convex; existence of the minimizer may not be guaranteed.

It is often convenient to transform the minimization problem (A.1) into a differential equation, called the *Euler-Lagrange* (EL) equation. Applying calculus of variations [21] to (A.1), the EL equation can be formulated as

$$-p \nabla \cdot \left( \frac{\nabla u}{|\nabla u|^{2-p}} \right) = 2\lambda (f - u).$$

Following the idea of Marquina and Osher [12], we multiply both sides of the above equation by  $|\nabla u|^{2-p}/p$  and then parameterize the energy descent direction by an artificial

time  $t$  to formulate a new evolutionary denoising model: Find  $u = u(\mathbf{x}, t)$ ,  $t > 0$ , by solving

$$\frac{\partial u}{\partial t} - |\nabla u|^q \nabla \cdot \left( \frac{\nabla u}{\|\nabla u\|^q} \right) = \beta (f - u), \quad (\text{A.2})$$

where  $u(\mathbf{x}, 0) = f(\mathbf{x})$ ,  $q = 2 - p$ , and  $\beta = 2\lambda|\nabla u|^q/p$ . Here  $\|\cdot\|$  is the same as  $|\cdot|$  in definition, but we write them separately just for convenience in numerical approximation. We are interested in the case  $1 < q < 2$  (non-convex) [9]. In this article, we call the above model the *enhanced total variation minimization* (ETVM).

In the next subsections, we present anisotropic numerical schemes for (A.2) which are conditionally stable and able to remove noise effectively with a minimum numerical dissipation.

## A.2. The time-stepping procedure

Denote the timestep size by  $\Delta t$ . Set  $t^n = n\Delta t$  and  $u^n = u(\cdot, t^n)$  for  $n \geq 0$ . Let  $\mathcal{A}^{n-1}$  be a linearized approximation of the diffusion part of the ETVM (A.2) in the  $n$ th level, i.e., for  $m = n - 1, n$ ,

$$\mathcal{A}^{n-1} u^m \approx -|\nabla u^{n-1}|^q \nabla \cdot \left( \frac{\nabla u^m}{\|\nabla u^{n-1}\|^q} \right). \quad (\text{A.3})$$

We may assume that  $\mathcal{A}^{n-1}$  is separable into locally one-dimensional problems:

$$\mathcal{A}^{n-1} = \mathcal{A}_1^{n-1} + \mathcal{A}_2^{n-1},$$

where  $\mathcal{A}_1^{n-1}$  and  $\mathcal{A}_2^{n-1}$  are tri-diagonal matrices. We will construct explicitly such a linearized algebraic system in the next subsection.

Let  $\mathcal{B}^{n-1} = \mathcal{A}^{n-1} + \beta I = \mathcal{B}_1^{n-1} + \mathcal{B}_2^{n-1}$ , where

$$\mathcal{B}_\ell^{n-1} = \mathcal{A}_\ell^{n-1} + \frac{1}{2}\beta I, \quad \ell = 1, 2.$$

Then, one can formulate an incomplete  $\theta$ -method for the ETVM (A.2):

$$\begin{aligned} \frac{u^n - u^{n-1}}{\Delta t} + \mathcal{B}^{n-1} [\theta u^n + (1 - \theta)u^{n-1}] \\ = \beta f, \quad 0 \leq \theta \leq 1, \end{aligned} \quad (\text{A.4})$$

and its associated alternating direction implicit (ADI) procedure [5], [6], [7], [16]:

$$\begin{aligned} [1 + \theta \Delta t \mathcal{B}_1^{n-1}] u^* \\ = [1 - (1 - \theta) \Delta t \mathcal{B}_1^{n-1} - \Delta t \mathcal{B}_2^{n-1}] u^{n-1} \\ + \Delta t \beta f, \\ [1 + \theta \Delta t \mathcal{B}_2^{n-1}] u^n = u^* + \theta \Delta t \mathcal{B}_2^{n-1} u^{n-1}, \end{aligned} \quad (\text{A.5})$$

where  $u^*$  is an intermediate solution. In this article, we call (A.5) the  $\theta$ -ADI.

## A.3. Anisotropic numerical schemes

We close this section by constructing  $\mathcal{A}^{n-1}$ , anisotropic numerical schemes for the diffusion part of the ETVM. For simplicity, we will focus on the construction of  $\mathcal{A}_1$ ; it is straightforward to apply the same schemes for  $\mathcal{A}_2$  with a concern on coordinate change.

Let  $\mathbf{x}_{ij}$  be the  $ij$ -th pixel in the image and  $u_{ij} = u(\mathbf{x}_{ij})$ . We will construct the row of  $\mathcal{A}_1^{n-1}$  corresponding to the pixel  $\mathbf{x}_{ij}$ ,  $[\mathcal{A}_1^{n-1}]_{ij}$ , which consists of three consecutive

non-zero elements which represent the connection of  $u_{ij}$  to  $u_{i-1,j}$  and  $u_{i+1,j}$ . We first define  $\mathcal{D}_x u_{i-1/2,j}^{n-1}$  as a finite difference approximation of  $\|\nabla u^{n-1}\|$  evaluated at  $\mathbf{x}_{i-1/2,j}$ , the mid point of  $\mathbf{x}_{i-1,j}$  and  $\mathbf{x}_{i,j}$ :

$$\begin{aligned} \mathcal{D}_x u_{i-1/2,j}^{n-1} &= \left( (u_{i,j}^{n-1} - u_{i-1,j}^{n-1})^2 \right. \\ &\left. + \left[ \frac{1}{2} \left( \frac{u_{i-1,j+1}^{n-1} + u_{i,j+1}^{n-1}}{2} - \frac{u_{i-1,j-1}^{n-1} + u_{i,j-1}^{n-1}}{2} \right) \right]^2 \right)^{1/2}, \end{aligned} \quad (\text{A.6})$$

and let

$$\begin{aligned} d_{ij,W}^{m-1} &= [(\mathcal{D}_x u_{i-1/2,j}^{n-1})^2 + \varepsilon^2]^{q/2}, \\ d_{ij,E}^{n-1} &= d_{i+1,j,W}^{n-1}, \end{aligned} \quad (\text{A.7})$$

where the regularization parameter  $\varepsilon > 0$  has been introduced to prevent the quantity from approaching zero. Then, the difference schemes for the diffusion part can be formulated as

$$\begin{aligned} \left( \frac{u_x^m}{\|\nabla u^{n-1}\|^q} \right)_x (\mathbf{x}_{ij}) &\approx \frac{1}{d_{ij,W}^{m-1}} u_{i-1,j}^m + \frac{1}{d_{ij,E}^{m-1}} u_{i+1,j}^m \\ &\quad - \left( \frac{1}{d_{ij,W}^{m-1}} + \frac{1}{d_{ij,E}^{m-1}} \right) u_{ij}^m, \\ |\nabla u^{n-1}|^q (\mathbf{x}_{ij}) &\approx 2 \frac{d_{ij,W}^{n-1} \cdot d_{ij,E}^{n-1}}{d_{ij,W}^{n-1} + d_{ij,E}^{n-1}}. \end{aligned} \quad (\text{A.8})$$

Thus the  $ij$ -th row of  $\mathcal{A}_1^{n-1}$  reads

$$[\mathcal{A}_1^{n-1}]_{ij} = (-a_{ij,W}^{n-1}, 2, -a_{ij,E}^{n-1}), \quad (\text{A.9})$$

where

$$a_{ij,W}^{n-1} = \frac{2 d_{ij,E}^{n-1}}{d_{ij,W}^{n-1} + d_{ij,E}^{n-1}}, \quad a_{ij,E}^{n-1} = \frac{2 d_{ij,W}^{n-1}}{d_{ij,W}^{n-1} + d_{ij,E}^{n-1}}. \quad (\text{A.10})$$

Note that  $a_{ij,W}^{n-1} + a_{ij,E}^{n-1} = 2$ .

The above anisotropic schemes incorporated with the  $\theta$ -ADI have been numerically verified to be efficient and reliable in image denoising, showing desirable properties in preserving edges. Of course, the parameters ( $q$ ,  $\Delta t$ ,  $\theta$ , and  $\beta$ ) must be chosen appropriately.