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TIME DISCRETIZATION OF TRANSITION LAYER DYNAMICS IN ONE-DIMENSIONAL VISCOELASTIC SYSTEMS*

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Abstract. We investigate how evolution occurs as the strain u_x of a viscoelastic system $u_{tt} - (\sigma(u_x) + u_{xt})_x + u = 0$ goes towards a state of equilibrium. The time limit of u_x eventually shows a finite number of discontinuous interfaces if the strain starts from the continuous initial data whose transition layers are steep enough and the initial energy is sufficiently small. The number of phases is conserved and the transition layers stay in the initial position of interfaces. The results are obtained by using the implicit time discretization method and the Andrews–Pego transformed equations.

Key words. implicit time discretization, viscous dissipation, transition layers, Andrews–Pego transformed equations, nonconvex energy

AMS subject classifications. 35G25, 74N25, 74D10

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Introduction. There are various results on the phase transitions of microstructured elastic crystals [2, 5, 12, 16, 25, 27, 30, 32, 33]. Nonconvex double-well free energy induces hysteretic behavior of the fine microstructures of the material. The usual approach involves the minimization of the elastic energy. Due to the nonconvexity of the free energy, every minimizing sequence fails to attain a minimizer and induces the formation of finer and finer oscillations of the sequence [5, 6, 30]. However, under the presence of the energy dissipation, such a behavior is prevented and the solution converges to the minimizer of the energy [3, 15, 27].

This article focuses on the one-dimensional viscoelastic system

$$u_{tt} - (\sigma(u_x) + u_{xt})_x + u = 0, \tag{0.1}$$

where u is a mapping from $(0, 1) \times (0, \infty) \subset \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ under appropriate initial and boundary conditions and $\sigma(x) = W'(x)$ for some stored energy function $W : \mathbb{R} \to \mathbb{R}$.

The system describes a time-dependent elastic bar with a nonconvex energy W and a viscous stress u_{xxt} with the zero displacement boundary conditions. The bar interacts with an elastic foundation u. In other words, the bar is placed on a system of linearly elastic springs [32].

Many global existence results for the solutions of similar systems are available [1, 3, 4, 7, 9, 11, 13, 14, 15, 16, 17, 18, 21, 24, 26, 27, 28]. The existence of a weak solution for the viscoelastic-type materials was developed for the cases without assuming the ellipticity of the free energy W [27], the convexity of W, or the Lipschitz continuity of σ [15]. In either case, the viscous dissipation plays a significant role in the strong convergence of the minimizing sequences. In the higher-dimensional case, Friesecke and Dolzmann [15] approached the results by an approximation, called the time discretized solution, on each sufficiently small time interval and using the Andrews–Pego transformed equations which were introduced in [1, 25].

The dynamics of the transition layers on the viscoelastic system (0.1) is the main topic in this paper. The transition layers are defined by the part of the strain u_x

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where the norm of u_x is less than a sufficiently small number. For the dynamics, we need the following two assumptions:

(I) The continuous initial data u_0 must have steep enough transition layers; that is, the norm of $(u_0)_{xx}$ should be sufficiently large at the position of transition layers of the initial data.

(II) The initial energy where the energy functional is defined by

$$E(u,v) = \int_0^1 \left[\frac{1}{2}(u(x))^2 + W((u_x(x)) + \frac{1}{2}(v(x))^2\right] dx$$

must be sufficiently small.

Under these assumptions, the time limit of u_x experiences a discontinuity at a finite number of points. More precisely, the transition layers get steeper and eventually become discontinuous at the time limit. Away from these finitely many points, the solution remains continuous and converges to a steady state. The number of transition layers remains the same and the transition layers of the solution are always within the intervals of initial layers, which is comparable to the results of stick-slip motion of layers in a system with the time-dependent displacement boundary conditions [33]. It was proven in [33] that the dynamics exhibits a different behavior than our main results. The layers do not stay in the initial intervals and will move both forward and backward.

Some important physical applications include a phase transformation in microstructured elastic crystals caused by changes in temperature, stresses, or incident electromagnetic waves. The nonconvex elastic energy functional induces finer and finer oscillations, but under the given initial state, the dynamics prevents the nucleation of more phases.

Friesecke and McLeod [16] proved the dynamics of transition layers employing the semigroup approach. In this paper, we use the method of time discretization to show the results. The time discretization theory has a long history [8, 10, 15, 19]. The scheme has been used for the nonlinear diffusion equations [8] and for the parabolic equations [19]. The second order time discretization on a related problem was first introduced in [10].

The existence of the solution was achieved in [16] by proving the existence of the Andrews–Pego transformations under an appropriate fractional power space. However, in [15], the variational approach was utilized for the proof of existence. It was proven that the time discretized functional

$$J^{m,j}[u] := \int_0^1 \left[\frac{1}{2m^2}|u - 2u^{m,j-1} + u^{m,j-2}|^2 + W(u_x) + \frac{1}{2m}|u_x - u_x^{m,j-1}|^2 + \frac{1}{2}|u|^2\right] dx$$

for each time interval $((j-1)m,\ jm],\ \ j \in \mathbb{N}$, where $m > 0$ is a fixed and sufficiently small time stepsize, has a minimizer $u^{m,j}$ which is a weak solution to the following discretized version of (0.1):

$$\text{(0.2)} \qquad \frac{1}{m^2}(u - 2u^{m,j-1} + u^{m,j-2}) - (\sigma(u_x))_x - \frac{1}{m}(u_x - u_x^{m,j-1})_x + u = 0$$

on $((j-1)m,\ jm],\ \ j \in \mathbb{N}$. The key idea of the approach is the following: The nonconvexity of the stored energy function W is compensated by the discretized viscous damping term $\frac{1}{2m}|u_x - u_x^{m,j-1}|^2$ to provide the convexity of the functional. It was also proven in [15] that the $W_0^{1,p}$ limit of the time discretized solutions is the weak solution of (0.1) as the time stepsize m approaches zero. The time discretization scheme naturally applies to the proof of the dynamics of the phase transition in this paper. The method is rather straightforward since it does not introduce any new space. There is nevertheless still a question whether the asymptotic behaviors of (0.1) and (0.2) commute or not. Is there any equivalence between $jm \to \infty$ for fixed m and then $m \to 0$ for the discretized problem, and $t \to \infty$ for the original problem? The question is discussed in the last section of the paper.

As in previous works [16, 27], the decay of the energy functional $E(u, v)$ is the crucial point of the proof. We prove in section 4 that the discretized energy functional is nonincreasing and bounded by the initial energy. A priori estimates are also proved in this section. We show next the existence and the equilibrium state as the limit $u_\star^m$ (as $j \to \infty$) of the discretized solution for fixed m in section 5. The main proof of the dynamics is conducted in section 6. We prove in this section that the transition layers approach a jump discontinuity as $j \to \infty$ by showing that a finite number of intervals where the strain is steep enough are decreasing to a finite number of isolated points. Unfortunately, the intervals in $(0, 1)$ where the norm of the strain $u_x^{m,j}$ is sufficiently small (denote the intervals as $I(u_x^{m,j})$) do not decrease monotonically as $j \to \infty$ in general. Instead, we introduce the time discretized version of Andrews–Pego transformed equations,

$$\begin{aligned} p^{m,j}(x) &:= \frac{1}{m}\int_0^x [u^{m,j}(y) - u^{m,j-1}(y)]dy - \frac{1}{m}\int_0^1\int_0^z [u^{m,j}(y) - u^{m,j-1}(y)]dydz, \\ q^{m,j}(x) &:= u_x^{m,j}(x) - p^{m,j}(x), \end{aligned}$$

and consider the finite number of intervals in $(0, 1)$ where the norm of $q^{m,j}$ is sufficiently small (denote them as $I(q^{m,j})$). We show that the $I(q^{m,j})$ decrease monotonically and exponentially to the isolated points as $j \to \infty$, and the intervals $I(u_x^{m,j})$ are contained in the $I(q^{m,j})$. The solution at the limit exhibits a jump discontinuity because of the decrease of the $I(q^{m,j})$ and the fact that the $I(u_x^{m,j})$ are contained in the $I(q^{m,j})$. In the last section, we summarize the relationship between the asymptotic behaviors of (0.1) and (0.2).

The interaction of the bar with an elastic foundation u induces a finely layered microstructure [12]. It has also been shown using the bifurcation analysis that the elastic foundation induces oscillations in the one-dimensional case of the static problem [32]. Nevertheless, under the assumption of low initial energy, the results still hold without the elastic foundation, and only minor change of the proof is required.

Another advantage is that the method is also useful in the practical implementation of the numerical solution of the system. The numerical results for the related problems have been discussed [4, 20, 29, 30, 31]. An implicit finite difference scheme for the homogeneous boundary conditions was achieved in [29], and the numerical methods for the time-dependent boundary conditions were investigated in [31]. In [20], the efficient numerical algorithms for the system in both one- and two-dimensional cases were developed. Applications to the microscale heat transfer equations will also appear in the near future.

The dynamics on similar problems was investigated in [22, 23]. The stability of the incompressible viscoelastic non-Newtonian fluid flows was observed in these papers. Investigating this type of spurt phenomena using the method of time discretization would be very interesting for future work.

1. The initial-boundary value problem and hypotheses. Consider the initial-boundary value problem

$$\begin{aligned} &u_{tt} - (\sigma(u_x) + u_{xt})_x + u = 0 \qquad (x \in (0,1), \quad t \in (0,\infty)), \\ &u|_{x=0} = u|_{x=1} = 0 \qquad (t \in [0,\infty)), \\ &u|_{t=0} = u_0, \quad u_t|_{t=0} = v_0 \qquad (x \in [0,1]), \end{aligned} \tag{1.1}$$

where u is a function from $(0,1) \times (0,\infty) \subset \mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, $\sigma = W'$, and W is a stored energy function satisfying the following conditions:

(H1) $W \in C^2(\mathbb{R}), W' = \sigma$.

(H2) There exist $c > 0,\ C > 0$, and $p \geq 2$ such that
$c|z|^p - C \leq W(z) \leq C(|z|^p + 1), \ \ |\sigma(z)| \leq C(|z|^{p-1} + 1)$ (coercivity).

(H3) W: double-well potential, that is, there exist $z_- < z_1 < 0 < z_2 < z_+$ such that $W(z_\pm) = 0, W > 0$ elsewhere, $W'(0) = 0, W''|_{(z_1,z_2)} < 0, W''|_{\mathbb{R}\setminus[z_1,z_2]} > 0$.

The stored energy function W is usually a fourth order nonconvex polynomial, and the most common example is $W(z) = \frac{1}{4}(z^2-1)^2$. Moreover, assume

(A1) (smoothness and a priori bounds) $u_0 \in C^2, v_0 \in W_0^{1,2}, ||(u_0)_x||_{L^\infty} + ||v_0||_{W^{1,2}} \leq M$;

(A2) (low initial energy) $E(u_0, v_0) < \epsilon$, where

$$E(u,v) := \int_0^1 \left(\frac{1}{2}u^2 + W(u_x) + \frac{1}{2}v^2\right) dx;$$

(A3) (no transition layers at $x = 0, 1$) $\mathcal{L}_\rho(0) := \{x \in [0,1] : |(u_0)_x(x)| \leq \rho\} \subset (0,1)$;

(A4) (steepness of transition layers) $|(u_0)_{xx}(x)| \geq K$ in $\mathcal{L}_\rho(0)$

for some $M, \epsilon, \rho, K > 0$. Here, $\epsilon,\ \rho$ are sufficiently small numbers and K is a sufficiently large number.

Let the connected components of $\mathcal{L}_\rho(0)$ be denoted by $[(a_0)_i, (b_0)_i]$, $i = 1, \ldots, N$ $(0 < (a_0)_1 < (b_0)_1 < \cdots < (a_0)_N < (b_0)_N < 1)$. Note that by assumption (A4), there exists only one zero of $(u_0)_x$ in each interval $[(a_0)_i, (b_0)_i]$. Let the zeros of $(u_0)_x$ be $(x_0)_i, (x_0)_i \in [(a_0)_i, (b_0)_i], i = 1, \ldots, N$.

2. The time discrete scheme for the solution. Let $m > 0,\ m \ll 1$ be the time stepsize of our problem. The m will be fixed throughout the paper except for the last section. Let $u^{m,0} := u_0, v^{m,0} := v_0, u^{m,-1} := u_0 - mv_0$. For each time interval $((j-1)m,\ jm], j \in \mathbb{N}$, define the following functional inductively:

$$J^{m,j}[u] := \int_0^1 \left[\frac{1}{2m^2}|u - 2u^{m,j-1} + u^{m,j-2}|^2 + W(u_x) + \frac{1}{2m}|u_x - u_x^{m,j-1}|^2 + \frac{1}{2}|u|^2\right] dx$$

on the Sobolev space $W_0^{1,p}((0,1),\mathbb{R})$, where p is the coercivity exponent of W in (H2). It was shown that $J^{m,j}$ attains a minimum $u^{m,j}$ if W satisfies the hypotheses (H1), (H2), and (H3) since the first and the fourth integrands are convex and the nonconvex term $W(u_x)$ combined with the viscous dissipation term $\frac{1}{2m}|u_x - u_x^{m,j-1}|^2$ provides the weakly lower semicontinuity [15]. It can be easily verified that for each time interval $((j-1)m,\ jm],\ j \in \mathbb{N},\ u^{m,j}(x)$ is the weak solution of

$$\frac{1}{m^2}(u - 2u^{m,j-1} + u^{m,j-2}) - (\sigma(u_x))_x - \frac{1}{m}(u_x - u_x^{m,j-1})_x + u = 0, \tag{2.1}$$

which is the time approximated equation of (1.1). The $u^{m,j}$ is thus called the time discretized solution of the problem (1.1). Assume that $u^{m,j}$ satisfies the boundary conditions of (1.1) for each $j \in \mathbb{N}$. On the time interval $((j-1)m,\ jm]$, $j \in \mathbb{N}$, we define the linear interpolation function $u^j(x,t)$ of $u^{m,j}(x)$ as follows:

$$u^j(x,t) := \left(\frac{mj-t}{m}\right) u^{m,j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) u^{m,j}(x). \tag{2.2}$$

It is now important to define the functions which are called the Andrews–Pego transformed equations. The equations will play a crucial role for the proof of the main results. Define

$$\begin{aligned} p_0(x) &:= \int_0^x v_0(y)dy - \int_0^1 \int_0^z v_0(y)dydz, \\ q_0(x) &:= (u_0)_x(x) - p_0(x), \\ p^{m,j}(x) &:= \frac{1}{m}\int_0^x [u^{m,j}(y) - u^{m,j-1}(y)]dy - \frac{1}{m}\int_0^1 \int_0^z [u^{m,j}(y) - u^{m,j-1}(y)]dydz, \\ q^{m,j}(x) &:= u_x^{m,j}(x) - p^{m,j}(x) \end{aligned}$$

for all $j \in \mathbb{N}$. Note that $p_x^{m,j}(x) = \frac{u^{m,j}(x)-u^{m,j-1}(x)}{m} =: v^{m,j}(x)$. For all $j \in \mathbb{N}$ and $(j-1)m < t \le jm$, define the interpolation functions of $p^j(x,t)$, $q^j(x,t)$, and $v^j(x,t)$ of $p^{m,j}(x)$, $q^{m,j}(x)$, and $v^{m,j}(x)$, respectively, in the same way as (2.2).

3. Main results. The following theorem describes the dynamical behavior of the transition layers.

THEOREM 3.1. *Suppose the stored energy function W and the initial data $(u_0,\ v_0) \in W_0^{1,\infty} \times L^2$ are assumed to satisfy* (H1)–(H3), (A1)–(A4). *Then the following hold:*

(P1) (Preservation of number of zeros.) *The number of zeros of $u_x^{m,j}$, denoted by $N(j)$, is finite, is positive for all $j \in \{0\} \cup \mathbb{N}$, and is independent of j. Let the zeros be denoted by $0 < x_1^m(j) < \cdots < x_N^m(j) < 1$.*

(P2) (Preservation of intervals of transition layers.) *The number of connected components of $\mathcal{L}_{\frac{\rho}{2}}(j) := \{x \in (0,1) : |u_x^j(x,t)| \le \frac{\rho}{2}\}$ is finite, is positive for all $j \in \mathbb{N}$, and is independent of j, and in each connected component, $u_x^j(x,t)$ is strictly monotone and has exactly one zero. Let the connected components of $\mathcal{L}_{\frac{\rho}{2}}(j)$ be denoted by $[a_i^m(j), b_i^m(j)]$, $i = 1, \ldots, N$ $(0 < a_1^m(j) < b_1^m(j) < \cdots < a_N^m(j) < b_N^m(j) < 1)$.*

(P3) (Lock-in and exponential steepening of transition layers.) *For all $j \in \mathbb{N}$, $i = 1, \ldots, N$, and for some $K_0 > 0$,*

$$x_i^m(j) \in [a_i^m(j), b_i^m(j)] \subset [(a_0)_i, (b_0)_i],$$

$$|u_{xx}^j(x,t)| \ge \frac{K_0}{2} e^{\sigma_0 jm} \quad \forall x \in \mathcal{L}_{\frac{\rho}{2}}(j) = \bigcup_{i=1}^N [a_i^m(j), b_i^m(j)],$$

$$|b_i^m(j) - a_i^m(j)| \le \frac{2\rho}{K_0} e^{-\sigma_0 jm},$$

where $\sigma_0 := \min_{[-\rho,\rho]} |\sigma'| > 0$.

(P4) (Convergence of phases.) $\lim_{j\to\infty} x_i^m(j) =: (x_\star)_i^m$ *exists for all $i = 1, \ldots, N$ and $(x_\star)_i^m \in [(a_0)_i, (b_0)_i]$ (in particular, $0 < (x_\star)_1^m < \cdots < (x_\star)_N^m < 1$).*

(P5) (Jump discontinuity of the limit state.) $\lim_{j\to\infty} u_x^{m,j} =: (u_\star^m)_x$ *(which exists as an L^p limit) is continuous on $(0,1)\backslash\{(x_\star)_1^m, \ldots, (x_\star)_N^m\}$ but discontinuous at every $(x_\star)_i^m$ for all $i = 1, \ldots, N$.*

4. Energy decay and a priori estimates. Let $t \in ((j-1)m,\ jm]$, $j \in \mathbb{N}$. We first prove the decay of the energy functional:

$$E(t) = E(u^j, v^j) = \int_0^1 \left[\frac{1}{2}(u^j(x,t))^2 + W(u_x^j(x,t)) + \frac{1}{2}(v^j(x,t))^2\right] dx.$$

LEMMA 4.1. *$E(t)$ is nonincreasing, bounded by the initial data on $((j-1)m,\ jm]$ for all $j \in \mathbb{N}$.*

Proof. Recall that $u^{m,j}$, $j \in \mathbb{N}$, satisfies (2.1). That is, the equation

$$v_t^j - \sigma(u_x^{m,j})_x - v_{xx}^{m,j} + u^{m,j} = 0 \tag{4.1}$$

is satisfied for all $j \in \mathbb{N}$. Then the following holds:

$$\begin{aligned}\frac{d}{dt}E(t) = \int_0^1 &[v^{m,j}\cdot v_t^j + \sigma(u_x^{m,j})\cdot u_{xt}^j + u^{m,j}\cdot u_t^j + (v^j - v^{m,j})v_t^j \\ &+ (\sigma(u_x^j) - \sigma(u_x^{m,j}))\cdot u_{xt}^j + (u^j - u^{m,j})\cdot u_t^j]dx \end{aligned}\tag{4.2}$$

$$\begin{aligned}= \int_0^1 &[v^{m,j}\{v_t^j - \sigma(u_x^{m,j})_x + u^{m,j}\} + (v^j - v^{m,j})v_t^j \\ &+ (\sigma(u_x^j) - \sigma(u_x^{m,j}))\cdot u_{xt}^j + (u^j - u^{m,j})\cdot u_t^j]dx \end{aligned}\tag{4.3}$$

$$\begin{aligned}= \int_0^1 &\Bigg[v^{m,j}\cdot v_{xx}^{m,j} + \frac{(t-jm)}{m^2}\cdot|v^{m,j} - v^{m,j-1}|^2 \\ &+ (\sigma(u_x^j) - \sigma(u_x^{m,j}))\cdot u_{xt}^j + \frac{(t-jm)}{m^2}\cdot|u^{m,j} - u^{m,j-1}|^2\Bigg]dx\end{aligned}$$

$$\begin{aligned}&= -\int_0^1 |v_x^{m,j}|^2 dx + \frac{(t-jm)}{m^2}\int_0^1 |v^{m,j} - v^{m,j-1}|^2 dx \\ &\quad + \int_0^1 (\sigma(u_x^j) - \sigma(u_x^{m,j}))\cdot u_{xt}^j dx + (t-jm)\int_0^1 |v^{m,j}|^2 dx\end{aligned}\tag{4.4}$$

for $(j-1)m < t \le jm$. The first three terms of (4.2) are the same as the first term of (4.3) by the integration by parts and the boundary conditions of (1.1). By using the mean value theorem, and by the fact that the function σ' is bounded below by a negative number, that is, for all $y \in \mathbb{R}$, $\sigma'(y) \ge -L$ for some $L > 0$, the integrand of the third term of (4.4) is estimated in the following way:

$$\begin{aligned}(\sigma(u_x^j) - \sigma(u_x^{m,j}))\cdot u_{xt}^j &\le \sigma'(c^*)(u_x^j - u_x^{m,j})\cdot u_{xt}^j \\ &= (jm-t)\{-\sigma'(c^*)\}\frac{(u_x^{m,j} - u_x^{m,j-1})^2}{m^2} \\ &\le m\cdot \max_{y\in\mathbb{R}}\{-\sigma'(y)\}\frac{(u_x^{m,j} - u_x^{m,j-1})^2}{m^2} \\ &= mL(v_x^{m,j})^2 \end{aligned}\tag{4.5}$$

for some c^* between u_x^j and $u_x^{m,j}$. Moreover, both the second and the fourth term of (4.4) are negative since $t - jm \le 0$. Now the following inequalities on the energy $E(t)$ are derived:

$$\begin{aligned}\frac{d}{dt}E(t) &\le (-1 + mL)\int_0^1 |v_x^{m,j}|^2 dx \\ &\le -\frac{1}{2}\|v_x^{m,j}\|_{L^2}^2\end{aligned}$$

for all $t \in ((j-1)m, jm]$. ∎

Note that by taking an integral from $(j-1)m$ to jm on both sides of the above inequality, we get

$$E(jm) - E((j-1)m) \leq -\frac{1}{2} m \|v_x^{m,j}\|_{L^2}^2,$$

and after taking a summation from $j = 1$ to $j = S$, the following estimate is established:

$$E(Sm) - E(0) \leq -\frac{m}{2} \sum_{j=1}^{S} \|v_x^{m,j}\|_{L^2}^2.$$

Therefore,

$$\frac{m}{2} \sum_{j=1}^{S} \|v_x^{m,j}\|_{L^2}^2 \leq E(0) - E(Sm) < E(0) < \epsilon \tag{4.6}$$

for all $S \in \mathbb{N}$. Next, we show the several estimates on the functions which will play a significant role for the proof of Theorem 3.1.

LEMMA 4.2. *The following a priori estimates hold:*

(a) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\leq jm} \|p^j(\cdot,t)\|_{L^\infty} \leq \eta$,

(b) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\leq jm} \|\pi_a(\int_0^x u^j(y,t)dy)\|_{L^\infty} \leq \sigma_0\eta$, *where* $\pi_a(f) := f - \int_0^1 f$,

(c) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\leq jm} \|q^j(\cdot,t)\|_{L^\infty} \leq \tilde{K}$,

(d) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\leq jm} \|u^j(\cdot,t)\|_{L^\infty} \leq \tilde{K}$,

(e) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\leq jm} |\int_0^1 \sigma(u_x^j(x,t))dx| \leq \sigma_0\eta$,

(f) $\sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\leq jm} \|v^j(\cdot,t)\|_{L^\infty} = \sup_{j\in\mathbb{N}} \sup_{(j-1)m<t\leq jm} \|p_x^j(\cdot,t)\|_{L^\infty} \leq \tilde{K}$

for some constants $\tilde{K}$, $\eta > 0$, $\eta \ll 1$.

Proof. Since $\int_0^1 p^j(x,t)dx = 0$, $p^j(x',t) = 0$ for some x' in $(0,1)$. Hence, the following holds:

$$|p^j(x,t)| = \left|\int_{x'}^{x} p_x^j(y,t)dy\right| \leq \left(\int_0^1 |p_x^j(y,t)|^2 dy\right)^{\frac{1}{2}}.$$

Since $\|p_x^j(\cdot,t)\|_{L^2} = \|v^j(\cdot,t)\|_{L^2} \leq \sqrt{E(u^j,v^j)} \leq \sqrt{E(u_0,v_0)} \leq \sqrt{\epsilon}$, (a) is accomplished by choosing $\eta > 0$ such that $\eta > \max\{\sqrt{\epsilon}, 2\sqrt{\epsilon}/\sigma_0, C_5\sqrt{\epsilon}/\sigma_0\}$, where C_5 will be chosen later.

Similarly,

$$\begin{aligned}
\left\|\int_0^x u^j(y,t)dy\right\|_{L^\infty} &= \left\|\int_0^x \int_0^y (p^j(z,t) + q^j(z,t))dzdy\right\|_{L^\infty} \\
&\leq \left\|\int_0^x (p^j(y,t) + q^j(y,t))dy\right\|_{L^2} \\
&= \|u^j(\cdot,t)\|_{L^2} \\
&\leq \sqrt{\epsilon} \leq \frac{\sigma_0\eta}{2},
\end{aligned}$$

which proves (b).

By using (4.1),

$$q_t^j = \frac{u_x^{m,j}(x) - u_x^{m,j-1}(x)}{m} - \int_0^x v_t^j + \int_0^1 \int_0^x v_t^j$$

$$= -\pi_a(\sigma(u_x^{m,j})) + \pi_a\left(\int_0^x u^{m,j}\right) \tag{4.7}$$

$$= -\sigma(p^{m,j} + q^{m,j}) + e_1^{m,j}, \tag{4.8}$$

where $e_1^{m,j} = \int_0^1 \sigma(u_x^{m,j}(x))dx + \pi_a(\int_0^x u^{m,j}(y)dy)$. From the hypotheses (H2) and (H3), $\sigma(z) \leq W(z) + C_1$ for some $C_1 > 0, z \in \mathbb{R}$. This and estimate (b) imply

$$|e_1^{m,j}| \leq \int_0^1 [|W(u_x^{m,j})| + C_1]dx + \sigma_0\eta \leq \epsilon + C_1 + \sigma_0\eta < C_2 \tag{4.9}$$

for some $C_2 > 0$. Since $\|p^j\|_{L^\infty} < \eta$, from (4.8), $q_t^j < 0$ when $q^j \geq K_1$ and $q_t^j > 0$ when $q^j \leq -K_1$ for some sufficiently large $K_1 > 0$. Hence, q^j is bounded. Let $\tilde{K} > \max\{\eta + K_1, K_2\}$, where K_2 will be chosen later. This completes the proof of (c).

Note that

$$\|u_x^j\|_{L^\infty} \leq \|p^j\|_{L^\infty} + \|q^j\|_{L^\infty} \leq \eta + K_1 < \tilde{K}. \tag{4.10}$$

Now, (d) clearly follows from (4.10).

Note that by (4.10),

$$|\sigma(u_x^j)| \leq C_3 \tag{4.11}$$

for some $C_3 > 0$. Since q_t^j satisfies (4.8), (4.11) combined with (4.9) implies that q_t^j is uniformly bounded for all $j \in \mathbb{N}$ in L^∞ norm. Also, note that

$$|\sigma'(u_x^j)| \leq C_4 \tag{4.12}$$

for some $C_4 > 0$. From the coercivity condition (H2) on W and σ and by estimate (4.10),

$$C_5 := \sup_{z \in [-\tilde{K}, \tilde{K}] \setminus \{z_-, z_+\}} \frac{|\sigma(z)|}{\sqrt{W(z)}}$$

is well defined and

$$\left| \int_0^1 \sigma(u_x^j(x,t))dx \right| \leq \|\sigma(u_x^j)\|_{L^2} \leq C_5 \left(\int_0^1 |W(u_x^j)| \right)^{\frac{1}{2}} < C_5\sqrt{\epsilon} \leq \sigma_0\eta,$$

which proves (e).

It will be shown next that $\|p_{xx}^j\|_{L^2}$ is uniformly bounded for all $j \in \mathbb{N}$ in order to prove (f).

Since

$$p_{xx}^j(x,t) = r^j(x,t) + s^j(x,t),$$

where

$$r^j(x,t) := \left(\frac{mj-t}{m}\right) p_t^{j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) p_t^j(x), \tag{4.13}$$
$$s^j(x,t) := \left(\frac{mj-t}{m}\right) q_t^{j-1}(x) + \left(\frac{t-m(j-1)}{m}\right) q_t^j(x),$$

and $\|q_t^j\|_{L^\infty}$ is uniformly bounded for all $j \in \mathbb{N}$, one would only need to show that $\|p_t^j\|_{L^2}$ is uniformly bounded for all $j \in \mathbb{N}$. By (4.7) and the identity $u_{xt}^j = p_{xx}^{m,j}$, p_t^j satisfies the following equation:

$$p_t^j = p_{xx}^{m,j} + \pi_a \left[\sigma(p^{m,j} + q^{m,j}) - \int_0^x \int_0^{x'} (p^{m,j} + q^{m,j})\right]. \tag{4.14}$$

Let $f(p^{m,j}) := -q_t^j$. Note that

$$\|f(p^{m,j})\|_{L^\infty} < M_1 \tag{4.15}$$

for some $M_1 > 0$ since q_t^j is uniformly bounded. From (4.14),

$$p^{m,j} - p^{m,j-1} = m\Delta p^{m,j} + mf(p^{m,j}),$$

which implies

$$\begin{aligned}
p^{m,j} &= \frac{p^{m,j-1}}{(1-m\Delta)} + \frac{m}{(1-m\Delta)} f(p^{m,j}) \\
&= \frac{1}{(1-m\Delta)} \left[\frac{p^{m,j-2}}{(1-m\Delta)} + \frac{m}{(1-m\Delta)} f(p^{m,j-1})\right] + \frac{m}{(1-m\Delta)} f(p^{m,j}) \\
&= \frac{p^{m,j-2}}{(1-m\Delta)^2} + m\left[\frac{f(p^{m,j-1})}{(1-m\Delta)^2} + \frac{f(p^{m,j})}{(1-m\Delta)}\right] \\
&\cdots \\
&= \frac{p_0}{(1-m\Delta)^j} + m\left[\frac{f(p^{m,1})}{(1-m\Delta)^j} + \cdots + \frac{f(p^{m,j})}{(1-m\Delta)}\right].
\end{aligned}$$

Therefore,

$$p_t^j = \frac{p^{m,j} - p^{m,j-1}}{m} = \frac{\Delta p_0}{(1-m\Delta)^j} + m\sum_{k=1}^{j-1} \frac{\Delta f(p^{m,k})}{(1-m\Delta)^{j+1-k}} + \frac{f(p^{m,j})}{(1-m\Delta)}.$$

By incorporating the inequality $\|\frac{\Delta}{(1-m\Delta)}\|_{L^2} \le 1$ and (4.15), the following inequalities occur:

$$\begin{aligned}
\|p_t^j\|_{L^2} &\le \|\Delta p_0\|_{L^2} + m\sum_{k=1}^{j-1} \left\|\frac{1}{(1-m\Delta)^{j-k}} \cdot \frac{\Delta}{(1-m\Delta)} \cdot f(p^{m,k})\right\|_{L^2} + \|f(p^{m,j})\|_{L^2} \\
&\le \|\Delta p_0\|_{L^2} + mM_1 \cdot \sum_{k=1}^{j-1} \frac{1}{(1-m\lambda_1)^{j-k}} + M_1 \\
&\le \|\Delta p_0\|_{L^2} + \frac{M_1}{\lambda_1} \cdot \left[\frac{1}{(1-m\lambda_1)^{j-1}} - 1\right] + M_1 \\
&\le \|\Delta p_0\|_{L^2} + \left(-\frac{1}{\lambda_1} + 1\right) \cdot M_1.
\end{aligned}$$

Here, $\lambda_1 < 0$ is the largest eigenvalue of Δ. Therefore, $\|p_t^j\|_{L^2}$ is uniformly bounded for all $j \in \mathbb{N}$ and $\|p_x^j\|_{L^\infty} < K_2$ for some $K_2 > 0$. Proof of Lemma 4.2 is now complete. ∎

Remark. One can see from the proofs of Lemmas 4.1 and 4.2 that the energy decay and a priori estimates are independent of m for sufficiently small $m > 0$.

5. Equilibrium state as the limit of the solution as $j \to \infty$. We now introduce the function φ, which is called the phase function. This function will play an important role in proving the equilibrium state of the solution at the limit as $j \to \infty$. Fix $r > 0$, $r \ll 1$ such that for $\lambda \in [-r, r]$, the equation $\sigma(z) = \lambda$ has three different solutions $z_1(\lambda) < z_2(\lambda) < z_3(\lambda)$. Define

$$\varphi(z) = \begin{cases} i, & z \in \displaystyle\bigcup_{\lambda \in [-r,\, r]} z_i(\lambda), \quad i = 1, 2, 3, \\ \infty & \text{elsewhere}. \end{cases}$$

The next proposition states that the discretized solution $u^{m,j}$ converges in $W_0^{1,p}$ to an equilibrium state as j goes to infinity.

Proposition 5.1. *Suppose* (H1)–(H3), (A1)–(A4) *hold. Then the solution* $(u^{m,j}, v^{m,j})$ *of* (2.1) *converges strongly in* $W_0^{1,p} \times L^2$ $(1 \le p < \infty)$ *to some equilibrium state* $(u_\star^m, 0) \in W_0^{1,\infty} \times L^2$ *as* $j \to \infty$.

Proof. The proof consists of several lemmas. The following lemma states that under some appropriate conditions on the elastic stress $\sigma(u_x^{m,j}(x)) - \int_0^x u^{m,j}$ and the phase function φ, the strain $u_x^{m,j}$ converges to an equilibrium state. We must be careful when choosing the pointwise representatives of $u_x^{m,j}$ since in the measure zero sets of $(0, 1)$, we never know the behavior of the strain $u_x^{m,j}$. It is important to choose a good representative so that the limit state is continuous except for the finitely many points which are the zeros of the limit state.

Lemma 5.2. *Assume there exists a full measure subset* $\tilde{\Omega} \in (0, 1)$ *(measure of* $\tilde{\Omega}$ *is* 1*) and pointwise representatives* $\bar{w}^{m,j}$ *of* $u_x^{m,j}$ *such that*

(B1) $\sigma(\bar{w}^{m,j}(x)) - \int_0^x u^{m,j} =: \lambda_j^m(x) \to \lambda^m$ *as* $j \to \infty$ *for some* $\lambda^m \in (-r, r)$ *and all* $x \in \tilde{\Omega}$;

(B2) $\lim_{j\to\infty} \varphi(\bar{w}^{m,j}(x))$ *exists and is finite for all* $x \in \tilde{\Omega}$.

Then $\lim_{j\to\infty} \bar{w}^{m,j}(x) =: \bar{w}^m(x)$ *exists for all* $x \in \tilde{\Omega}$. *Moreover, the equivalence class* $\hat{w}^m$ *of* $\bar{w}^m$ *satisfies*

$$\|\hat{w}^m\|_{L_\infty} \le \tilde{K} \qquad \textit{and} \qquad u_\star^m(x) := \int_0^x \hat{w}^m \quad \textit{is in} \quad W_0^{1,\infty}.$$

Also,

$$\sigma((u_\star^m)_x(x)) - \int_0^x u_\star^m \equiv \lambda^m \ \ a.e., \qquad \varphi((u_\star^m)_x(x)) = \lim_{j\to\infty} \varphi(u_x^{m,j}(x)) \ \ a.e.$$

and

$$u^{m,j} \to u_\star^m \ \ in \ \ W_0^{1,p} \qquad (1 \le p < \infty).$$

Proof. Recall that $\sup_{j\in\mathbb{N}} \|u_x^j\|_{L^\infty} < \tilde{K}$ by (4.10). Define

$$\chi_i^{m,j}(x) := \begin{cases} 1, & x \in \tilde{\Omega} \text{ and } \varphi(\bar{w}^{m,j}(x)) = i \in \{1, 2, 3\}, \\ 0 & \text{else}, \end{cases}$$

$$\chi_\infty^{m,j}(x) := \begin{cases} 1, & x \in \tilde{\Omega} \text{ and } \varphi(\bar{w}^{m,j}(x)) = \infty, \\ 0 & \text{else.} \end{cases}$$

Let $x \in \tilde{\Omega}$. Since $\bar{w}^{m,j}(x) = z_i(\int_0^x u^{m,j} + \lambda_j^m(x))$ and $\chi_\infty^{m,j}(x) = 0$ if $\varphi(\bar{w}^{m,j}(x)) = i$, $i = 1, 2, 3$, the following equation holds:

$$\begin{aligned} &\bar{w}^{m,j}(x) - \bar{w}^{m,k}(x) \\ &= \sum_{i=1}^{3} \left[\chi_i^{m,j}(x) \cdot z_i \left(\int_0^x u^{m,j} + \lambda_j^m(x) \right) - \chi_i^{m,k}(x) \cdot z_i \left(\int_0^x u^{m,k} + \lambda_k^m(x) \right) \right] \\ &\quad + \chi_\infty^{m,j}(x) \cdot \bar{w}^{m,j}(x) - \chi_\infty^{m,k}(x) \cdot \bar{w}^{m,k}(x). \end{aligned} \tag{5.1}$$

Note that since $1 = \frac{d}{d\lambda^m}(\sigma(z_i(\lambda^m))) = \sigma'(z_i(\lambda^m)) \cdot z_i'(\lambda^m)$,

$$\begin{aligned} |z_i(a) - z_i(b)| &\leq \sup_{x \in [-r,\, r]} |z_i'(x)| \cdot |a - b| \\ &= \sup_{x \in [-r,\, r]} \frac{1}{|\sigma'(z_i(x))|} \cdot |a - b| \leq \frac{1}{\bar{M}} |a - b|, \end{aligned} \tag{5.2}$$

where $\bar{M} := \min_{z \in \sigma^{-1}([-r,r])} |\sigma'(z)|$. Let $\xi_{j,k}^m(x) = \int_0^x |u^{m,j} - u^{m,k}|$, $j, k \in \mathbb{N}$. Then

$$\begin{aligned} 0 \leq \frac{d}{dx}\xi_{j,k}^m(x) &= \left| \int_0^x (u_x^{m,j} - u_x^{m,k}) \right| \\ &= \left| \int_0^x \left[\sum_{i=1}^{3} \left\{ \chi_i^{m,j}(x') \cdot z_i \left(\int_0^{x'} u^{m,j} + \lambda_j^m(x') \right) \right.\right.\right. \\ &\qquad\qquad \left. - \chi_i^{m,k}(x') \cdot z_i \left(\int_0^{x'} u^{m,k} + \lambda_k^m(x') \right) \right\} \\ &\qquad \left.\left. + \chi_\infty^{m,j}(x') \cdot \bar{w}^{m,j}(x') - \chi_\infty^{m,k}(x') \cdot \bar{w}^{m,k}(x') \right] dx' \right| \\ &= \left| \int_0^x \left[\sum_{i=1}^{3} \chi_i^{m,k}(x') \{ \bar{w}^{m,j}(x') - \bar{w}^{m,k}(x') \} + \sum_{i=1}^{3} \{ \chi_i^{m,j}(x') - \chi_i^{m,k}(x') \} \bar{w}^{m,j}(x') \right.\right. \\ &\qquad \left.\left. + \chi_\infty^{m,j}(x') \cdot \bar{w}^{m,j}(x') - \chi_\infty^{m,k}(x') \cdot \bar{w}^{m,k}(x') \right] dx' \right| \end{aligned} \tag{5.3}$$

holds for $j, k \in \mathbb{N}$. Note that

$$\int_0^x \left[\sum_{i=1}^{3} (\chi_i^{m,j} - \chi_i^{m,k}) \bar{w}^{m,j} \right] \leq 2\tilde{K} |\{ x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) \neq \varphi(\bar{w}^{m,k}(x)) \}|$$

and

$$\begin{aligned} \int_0^x \left(\chi_\infty^{m,j} \cdot \bar{w}^{m,j} - \chi_\infty^{m,k} \cdot \bar{w}^{m,k} \right) &\leq \tilde{K} |\{ x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) \neq \varphi(\bar{w}^{m,k}(x)) \}| \\ &\quad + 2\tilde{K} |\{ x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) = \varphi(\bar{w}^{m,k}(x)) = \infty \}|. \end{aligned}$$

Therefore, the last three terms in (5.3) are dominated by

$$\begin{aligned} &3\tilde{K} |\{ x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) \neq \varphi(\bar{w}^{m,k}(x)) \}| \\ &+ 2\tilde{K} |\{ x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) = \varphi(\bar{w}^{m,k}(x)) = \infty \}| =: \delta_{j,k}^m. \end{aligned}$$

Let $x \in (0,1)$ be fixed. By assumption (B2), $\varphi(\bar{w}^{m,j}(x)) = \varphi(\bar{w}^{m,k}(x)) = i(x)$ for some $i(x) = 1,2,3$ if j,k are sufficiently large. Therefore, $\delta^m_{j,k} \to 0$ as $\min\{j,k\} \to \infty$. For each $i \in \{1,2,3\}$,

$$\int_0^x \chi_i^{m,k}(x')(\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x'))dx' \le \int_{J_1(i)} |\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x')|dx' + \int_{J_2(i)} |\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x')|dx',$$

where

$$J_1(i) := \{x' \in (0,x) : \chi_i^{m,j}(x') = \chi_i^{m,k}(x') = 1\},$$

$$J_2(i) := \{x' \in (0,x) : \chi_i^{m,j}(x') = 0,\ \chi_i^{m,k}(x') = 1\}.$$

In the set $J_1(i)$,

$$\begin{aligned}
|\bar{w}^{m,j}(x') - \bar{w}^{m,k}(x')| &= \left| z_i\left(\int_0^{x'} u^{m,j}(s)ds + \lambda_j^m(x')\right) - z_i\left(\int_0^{x'} u^{m,k}(s)ds + \lambda_k^m(x')\right)\right| \\
&\le \frac{1}{\bar{M}}\left[\int_0^{x'} |u^{m,j}(s) - u^{m,k}(s)|ds + |\lambda_j^m(x') - \lambda_k^m(x')|\right] \\
&= \frac{1}{\bar{M}}[\xi^m_{j,k}(x') + |\lambda_j^m(x') - \lambda_k^m(x')|].
\end{aligned}$$

Note that $J_2(i) \subset \{x \in (0,1) : \varphi(\bar{w}^{m,j}(x)) \ne \varphi(\bar{w}^{m,k}(x))\}$, $i = 1,2,3$. Hence,

$$\begin{aligned}
\frac{d}{dx}\xi^m_{j,k}(x) &\le 2\cdot\delta^m_{j,k} + \frac{1}{\bar{M}}\int_0^1 |\lambda_j^m(x') - \lambda_k^m(x')|dx' + \int_0^x \frac{1}{\bar{M}}\xi^m_{j,k}(x')dx' \\
&\le 2\cdot\delta^m_{j,k} + \frac{1}{\bar{M}}\|\lambda_j^m - \lambda_k^m\|_{L^1} + \frac{1}{\bar{M}}\xi^m_{j,k}(x) \\
&\le \epsilon^m_{j,k} + \frac{1}{\bar{M}}\xi^m_{j,k}(x),
\end{aligned}$$

where $\epsilon^m_{j,k} := 2\cdot\delta^m_{j,k} + \frac{1}{\bar{M}}\|\lambda_j^m - \lambda_k^m\|_{L^1}$. By assumption (B1), $\epsilon^m_{j,k} \to 0$ as $\min\{j,k\} \to \infty$. By Gronwall's inequality,

$$\xi^m_{j,k}(x) \le \epsilon^m_{j,k}\cdot\bar{M}\cdot\left(\exp\left(\frac{1}{\bar{M}}x\right) - 1\right) \to 0 \text{ as } \min\{j,k\} \to \infty.$$

Therefore,

$$\left|\int_0^x u^{m,j} - \int_0^x u^{m,k}\right| \le \xi^m_{j,k}(x) \to 0$$

as $\min\{j,k\} \to \infty$. By combining this with assumption (B1), we get

$$\left|\left(\int_0^x u^{m,j} + \lambda_j^m(x)\right) - \left(\int_0^x u^{m,k} + \lambda_k^m(x)\right)\right| \to 0 \text{ a.e.}$$

as $\min\{j,k\} \to \infty$. By assumption (B2), $\chi_i^{m,j}(x) = \chi_i^{m,k}(x) = 1$ for some $i(x) = 1,2,3$, and $\chi_\infty^{m,j}(x) = \chi_\infty^{m,k}(x) = 0$ for sufficiently large $j,k \in \mathbb{N}$. This implies that the right-hand side of (5.1) converges to zero, and thus $\bar{w}^{m,j}(x) - \bar{w}^{m,k}(x) \to 0$ for all $x \in \tilde{\Omega}$ as $\min\{j,k\} \to \infty$. Hence,

$$\lim_{j\to\infty} \int_0^x u^{m,j} =: U^m \quad \text{exists} \; \forall \, x \in [0,1],$$
$$\lim_{j\to\infty} \bar{w}^{m,j} =: \bar{w}^m \quad \text{exists} \; \forall \, x \in \tilde{\Omega}.$$

This implies that $u_x^{m,j}$ converges to the equivalence class $\hat{w}^m$ of $\bar{w}^m$ in L^1. Let $u_\star^m(x) := \int_0^x \hat{w}^m$. Since

$$u_\star^m(1) = \int_0^1 \hat{w}^m = \lim_{j\to\infty} \int_0^1 u_x^{m,j} = \lim_{j\to\infty} (u^{m,j}(1) - u^{m,j}(0)) = 0,$$

$u_\star^m$ satisfies the boundary conditions of (1.1). Moreover, since

$$u^{m,j}(x) = \int_0^x u_x^{m,j} \to \int_0^x \hat{w}^m = u_\star^m(x) \;\text{ in }\; C([0,1]),$$

$U^m = \int_0^x u_\star^m$. Hence,

$$u^{m,j} \to u_\star^m \;\text{ in }\; W^{1,p}, \qquad 1 \le p < \infty.$$

Since $u^{m,j}$, $u_x^{m,j}$ are uniformly bounded, $u_\star^m \in W_0^{1,\infty}$. Therefore,

$$u_x^{m,j} \to (u_\star^m)_x \quad \text{boundedly a.e.}$$

Since $\sigma(u_x^{m,j}) - \int_0^x u^{m,j} \to \sigma((u_\star^m)_x) - \int_0^x u_\star^m$ boundedly a.e. by assumption (B1),

$$\lambda^m = \sigma((u_\star^m)_x) - \int_0^x u_\star^m \quad \text{a.e.} \tag{5.4}$$

Since $(u_\star^m)_x$ lies in one of the three intervals $\bigcup_{\lambda\in[-r,\,r]} z_i(\lambda),\;\; i \in \{1,2,3\}$ a.e., we can choose the nice pointwise representatives $\bar{w}^{m,j}$ of $u_x^{m,j}$ such that (5.4) holds for the set $(0,1)$ except for the finitely many points which are the limits $(x_\star)_i^m$ of finitely many zeros $x_i^m(j), i = 1,\dots,N$, of $u_x^{m,j}$ in (P4). Hence, we can conclude that $(u^{m,j}, v^{m,j})$ converges to an equilibrium state $(u_\star^m, 0)$ strongly in $W_0^{1,p} \times L^2$. □

In the lemmas to follow, we will show that under the low initial energy, the assumptions (B1) and (B2) are satisfied. Lemma 5.3 shows that the convergence of mean elastic stress $\int_0^1 (\sigma(u_x^{m,j}) - \int_0^x u^{m,j}) dx$ implies the convergence of elastic stress $\sigma(u_x^{m,j}) - \int_0^x u^{m,j}$.

LEMMA 5.3. *Let $u^{m,j}$, $j \in \mathbb{N}$, be a solution of* (2.1). *Assume that*

$$\lim_{j\to\infty} \int_0^1 \left(\sigma(u_x^{m,j}) - \int_0^x u^{m,j} \right) dx =: \lambda^m \quad \textit{exists.}$$

Then

$$\sigma(u_x^{m,j}) - \int_0^x u^{m,j} \to \lambda^m \;\; \textit{a.e. as} \;\; j \to \infty.$$

Proof. By (4.7), the sufficient condition to the conclusion is when q_t^j goes to zero a.e. as $j \to \infty$. Define the following modification of the energy functional $E(t)$:

$$\widetilde{E}(t) := \int_0^1 \left[W(u_x^j(x,t)) + \frac{1}{2}(u^j(x,t))^2 + p^j(x,t) s^j(x,t) \right] dx,$$

where $s^j(x,t)$ is the interpolation function defined in (4.13). Note that $\widetilde{E}(t)$ is uniformly bounded and, moreover, sufficiently small since the first two terms are the part of energy functional $E(t)$ and the third term is small since $p^j(x,t)$ is small enough by estimate (a) of Lemma 4.2 and $s^j(x,t)$ is uniformly bounded since q_t^j is uniformly bounded for all $j \in \mathbb{N}$. By (4.1) and the integration by parts,

$$\begin{aligned}
\frac{d}{dt}\widetilde{E}(t) &= \int_0^1 [\,\sigma(u_x^j)\cdot u_{xt}^j + u^j\cdot u_t^j + p_t^j\cdot q_t^j + p_t^j(s^j - q_t^j) + p^j\cdot s_t^j]dx \\
&= \int_0^1 \Bigg[(\sigma(u_x^j) - \sigma(u_x^{m,j}))\cdot u_{xt}^j + (u^j - u^{m,j})\cdot u_t^j \\
&\qquad + \sigma(u_x^{m,j})\cdot u_{xt}^j + u^{m,j}\cdot u_t^j + u_{xt}^j\cdot q_t^j - (q_t^j)^2 \\
&\qquad + p_t^j\left(\left(\frac{mj-t}{m}\right)q_t^{m,j-1} + \left(\frac{t-m(j-1)-m}{m}\right)q_t^{m,j}\right) + p^j s_t^j\Bigg]dx \\
&\le mL\|v_x^{m,j}\|_{L^2}^2 + (t-mj)\|v^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx \\
&\quad + \int_0^1 \Bigg[(v_{xx}^{m,j} - v_t^j)\cdot u_t^j + u_{xt}^j u_{xt}^j - u_{xt}^j p_t^j \\
&\qquad + p_t^j\left(\frac{t-mj}{m}\right)\cdot(q_t^{m,j} - q_t^{m,j-1}) + p^j s_t^j\Bigg]dx \qquad (5.5)\\
&\le mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx \\
&\quad + \int_0^1 [-u_{xt}^j v_x^{m,j} - u_t^j v_t^j + u_{xt}^j u_{xt}^j + u_t^j p_{xt}^j + p_t^j\cdot(t-mj)\cdot s_t^j + p^j s_t^j]dx \qquad (5.6)\\
&= mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx + \int_0^1 s_t^j\Bigg[(t-mj)\cdot\left(\frac{p^{m,j} - p^{m,j-1}}{m}\right) \\
&\quad + \left(\frac{mj-t}{m}\right)p^{m,j-1} + \left(\frac{t-m(j-1)}{m}\right)p^{m,j}\Bigg]dx \\
&= mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx + 2\cdot\int_0^1 s_t^j\cdot p^j dx - \int_0^1 s_t^j\cdot p^{m,j} dx.
\end{aligned}$$

The first term of (5.5) is followed from estimate (4.5). The first four terms of the integrand of the third term of (5.6) vanish because of the identities $u_{xt}^j = v_x^{m,j}$, $p_{xt}^j = v_t^j$. Since

$$\begin{aligned}
\left|\int_0^1 s_t^j\cdot p^{m,j} dx\right| &\le \|p_{xx}^{m,j}\|_{L^2}\cdot\|s_t^j\|_{L^2} \\
&= \|v_x^{m,j}\|_{L^2}\cdot\left\|\frac{q_t^j - q_t^{j-1}}{m}\right\|_{L^2} \\
&\le \|v_x^{m,j}\|_{L^2}\cdot\left(\left\|\pi_a\left(\sigma'(c^{**})\cdot\frac{u_x^{m,j} - u_x^{m,j-1}}{m}\right)\right\|_{L^2} + \left\|\frac{u_x^{m,j} - u_x^{m,j-1}}{m}\right\|_{L^2}\right) \\
&\le M_2\cdot\|v_x^{m,j}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}\left|\int_0^1 s_t^j \cdot p^j\,dx\right| &\leq \|v_x^j\|_{L^2} \cdot \|s_t^j\|_{L^2}\\ &\leq M_3 \cdot (\|v_x^{m,j-1}\|_{L^2} + \|v_x^{m,j}\|_{L^2}) \cdot \|v_x^{m,j}\|_{L^2},\end{aligned}$$

and

$$\|v_x^{m,j-1}\|_{L^2} \cdot \|v_x^{m,j}\|_{L^2} \leq M_4 \cdot (\|v_x^{m,j-1}\|_{L^2}^2 + \|v_x^{m,j}\|_{L^2}^2)$$

for some c^{**} between $u_x^{m,j}$ and $u_x^{m,j-1}$, M_2, M_3, and $M_4 > 0$, the following estimate on $\frac{d}{dt}\widetilde{E}(t)$ holds:

$$\frac{d}{dt}\widetilde{E}(t) \leq mL\|v_x^{m,j}\|_{L^2}^2 - \int_0^1 (q_t^j)^2 dx + M_5 \cdot (\|v_x^{m,j-1}\|_{L^2}^2 + \|v_x^{m,j}\|_{L^2}^2) \tag{5.7}$$

for some $M_5 > 0$. By taking an integral from $(j-1)m$ to jm on both sides of (5.7), we get

$$\widetilde{E}(jm) - \widetilde{E}((j-1)m) \leq -m\int_0^1 (q_t^j)^2 + (mL + M_5)m\|v_x^{m,j}\|_{L^2}^2 + mM_5\|v_x^{m,j-1}\|_{L^2}^2.$$

By taking the summation $j = 1, \ldots, S$, we get the following estimate:

$$\widetilde{E}(Sm) - \widetilde{E}(0) \leq -m\sum_{j=1}^{S}\int_0^1 (q_t^j)^2 + (mL + M_6)\sum_{j=1}^{S} m\|v_x^{m,j}\|_{L^2}^2 + mM_5\|(v_0)_x\|_{L^2}^2$$

for some $M_6 > 0$. By (4.6),

$$\begin{aligned} m\sum_{j=1}^{S}\int_0^1 (q_t^j)^2 &\leq \widetilde{E}(0) - \widetilde{E}(Sm) + 2(mL + M_6)\epsilon + M_5 m\|(v_0)_x\|_{L^2}^2\\ &\leq |\widetilde{E}(0)| + |\widetilde{E}(Sm)| + 2(mL + M_6)\epsilon + \epsilon_1\\ &\leq \delta\end{aligned}$$

for some $\epsilon_1, \delta \ll 1$. Therefore, $m\sum_{j=1}^{\infty}\int_0^1 (q_t^j)^2 \leq \delta$, and this implies $q_t^j \to 0$ a.e. as $j \to \infty$. ∎

The next lemma shows the convergence of the phase function under the assumptions of the low initial energy and the convergence of mean elastic stress.

LEMMA 5.4. *Let $u^{m,j}$, $j \in \mathbb{N}$, be the solution of* (2.1). *Then the assumption* (B2) *in Lemma* 5.2 *holds.*

Proof. By (b) and (e) of Lemma 4.2, mean elastic stress $\int_0^1 \left(\sigma(u_x^{m,j}) - \int_0^x u^{m,j}\right) dx$ is sufficiently small. Then by Lemma 5.3, $\limsup_{j\to\infty}\left|\sigma(u_x^{m,j}) - \int_0^x u^{m,j}\right|$ is sufficiently small a.e. Combining this and estimate (b) of Lemma 4.2, $\limsup_{j\to\infty}\left|\sigma(u_x^j)\right| \leq \frac{2r}{3}$ a.e. This implies that for almost every x, there exists $J(x) \in \mathbb{N}$ such that

$$\{u_x^j(x,t) : j \geq J(x)\} \subseteq \sigma^{-1}([-r,\ r]) = \bigcup_{i=1}^{3}\bigcup_{\lambda\in[-r,\ r]} z_i(\lambda).$$

Since $\{u_x^j(x,t) : j \geq J(x)\}$ is connected, $u_x^j(x,t) \in \bigcup_{\lambda\in[-r,\ r]} z_i(\lambda)$ for all $j \geq J(x)$ and for some $i(x) = 1, 2$, or 3. This implies that $\lim_{j\to\infty}\varphi(u_x^j(x,t))$ exists and is finite a.e. Consequently, $\lim_{j\to\infty}\varphi(u_x^{m,j}(x))$ also exists and is finite a.e. ∎

The next lemma shows the convergence of mean elastic stress.

LEMMA 5.5. *Let* $u^{m,j}$ *be the solution of* (2.1). *Then*

$$\lim_{j\to\infty} \underbrace{\left[\int_0^1 \left(\sigma(u_x^{m,j}) - \int_0^x u^{m,j}\right) dx\right]}_{=:\, c(j)} \quad exists.$$

Proof. Suppose this fails. Then there exists a subsequence $j_k \to \infty$ such that $c(j_k) \to \lambda^m$ and another subsequence $j_s \to \infty$ such that $c(j_s) \to \bar{\lambda}^m$ for some $\lambda^m,\ \bar{\lambda}^m \in [-\frac{2r}{3}, \frac{2r}{3}]$ and $\lambda^m < \bar{\lambda}^m$. Then by Lemma 5.3, $\sigma(u_x^{m,j_k}) - \int_0^x u^{m,j_k} \to \lambda^m$ a.e. as $j_k \to \infty$ and $\sigma(u_x^{m,j_s}) - \int_0^x u^{m,j_s} \to \bar{\lambda}^m$ a.e. as $j_s \to \infty$. Also by Lemma 5.4, $\lim_{j_k\to\infty} \varphi(u_x^{m,j_k})$, $\lim_{j_s\to\infty} \varphi(u_x^{m,j_s})$ exist and are finite, respectively. Hence, these satisfy the assumptions (B1), (B2) of Lemma 5.2, and therefore there exist $u^m,\ \bar{u}^m \in W^{1,\infty}$ such that

$$\sigma(u_x^m)_x - \int_0^x u^m \equiv \lambda^m \ \text{ a.e.}, \quad \sigma(\bar{u}_x^m)_x - \int_0^x \bar{u}^m \equiv \bar{\lambda}^m \ \text{ a.e.},$$

$$\varphi(u_x^m(x)) = \lim_{j_k\to\infty} \varphi(u_x^{m,j_k}(x)) \ \text{ a.e.}, \quad \text{and} \quad \varphi(\bar{u}_x^m(x)) = \lim_{j_s\to\infty} \varphi(u_x^{m,j_s}(x)) \ \text{ a.e.}$$

Note that $\varphi(u_x^m(x)) = \varphi(\bar{u}_x^m(x)) =: \varphi_\infty(x)$ since the limit of the phase function is independent of λ^m and $\bar{\lambda}^m$.

Consider the case where $\varphi_\infty(x) \in \{1,3\}$ a.e. That is, the measure of the set $\Omega_u := \{x \in (0,1) : \varphi_\infty(x) = 2\}$ is zero. Now we introduce the following principle, whose proof was done in [16].

Comparison principle for weak solutions of the ordinary differential equation $\boldsymbol{\sigma(u_x)_x = u}$. Assume that $u,\ \bar{u} \in W^{1,\infty}$ satisfy

$$\sigma(u_x)_x - \int_0^x u \equiv \lambda \ \text{ a.e.}, \quad \sigma(\bar{u}_x)_x - \int_0^x \bar{u} \equiv \bar{\lambda} \ \text{ a.e.},$$

$\lambda < \bar{\lambda}$, $u(0) = \bar{u}(0) = 0$, $\sigma(u_x)$, $\sigma(\bar{u}_x) \in [-r, r]$ a.e., $\varphi(u_x) = \varphi(\bar{u}_x)$ a.e., and $\varphi(u_x) \in \{1,3\}$ a.e. Then $u(x) < \bar{u}(x)$ for all $x \in (0,1]$.

Since u^m and $\bar{u}^m$ satisfy the assumptions of the above comparison principle, $u^m(1) < \bar{u}^m(1)$. This contradicts the boundary conditions of (1.1). In the case when the measure of Ω_u is not zero, contradiction arises from the following modified principle, which was also proven in [16].

Refined comparison principle for weak solutions of the ordinary differential equation $\boldsymbol{\sigma(u_x)_x = u}$. Under the same assumptions as the comparison principle, but with the condition $\varphi(u_x) \in \{1,3\}$ a.e. replaced by $\int_0^1 W(u_x) < \epsilon$ and $|\Omega_u| \neq 0$, the inequality

$$u(1) < \bar{u}(1)$$

holds. ∎

Now Proposition 5.1 is complete. ∎

6. Dynamical behavior of the transition layers. If the set

$$\mathcal{L}_{\frac{\rho}{2}}(j) = \left\{x \in (0,1) : |u_x^j(x,t)| \le \frac{\rho}{2}\right\}$$

is monotonically decreasing to the finitely many isolated points as $j \to \infty$, we obtain the desired conclusion, since this is equivalent to the fact that the layers get steeper and eventually become discontinuous as j approaches infinity. However, the set $\mathcal{L}_{\frac{\rho}{2}}(j)$ is not decreasing as $j \to \infty$. We define the following set $\widetilde{\mathcal{L}}(j)$ instead and show that the set $\mathcal{L}_{\frac{\rho}{2}}(j)$ is contained in $\widetilde{\mathcal{L}}(j)$. We will then show that the set $\widetilde{\mathcal{L}}(j)$ is decreasing to the finitely many isolated points. Let $\eta \in (0, \frac{\rho}{4})$. Set $\rho_0 := \rho - \eta$. Define

$$\widetilde{\mathcal{L}}(j) := \{x \in (0,1) : |q^j(x,t)| \le \rho_0\}.$$

The following lemma states that the set of transition layers are always in the set $\widetilde{\mathcal{L}}(j)$ and furthermore in the set of initial transition layers $\mathcal{L}_\rho(0)$. This lemma plays an important role in showing the preservation of the number of transition layers.

LEMMA 6.1.

$$\mathcal{L}_{\frac{\rho}{2}}(j) \subseteq \widetilde{\mathcal{L}}(j) \subseteq \mathcal{L}_\rho(0) \quad \forall j \in \mathbb{N}.$$

Proof. If $x \in \mathcal{L}_{\frac{\rho}{2}}(j)$, then $|u_x^j(x,t)| \le \frac{\rho}{2}$. Therefore, by estimate (a) of Lemma 4.2,

$$|q^j(x,t)| = |u_x^j - p^j| \le \frac{\rho}{2} + \eta < \frac{\rho}{2} + \frac{\rho}{4} < \rho - \eta = \rho_0.$$

Now, $\widetilde{\mathcal{L}}(j) \subseteq \mathcal{L}_\rho(0)$ clearly follows. ∎

Next we show that the set $\widetilde{\mathcal{L}}(j)$ is exponentially decreasing to the finitely many isolated points.

LEMMA 6.2. *Assume $K > 4\tilde{K}$. Then for all $j \in \mathbb{N}$ and for some $C_0 > 0$,*

(i) $|q_x^j(x,t)| \ge C_0 e^{jm\sigma_0} |(q_0)_x|$ *if* $x \in \widetilde{\mathcal{L}}(j)$ *(exponential growth),*

(ii) $\widetilde{\mathcal{L}}(j+1) \subseteq \widetilde{\mathcal{L}}(j)$ *(monotonicity).*

Proof. We will show (i) by induction. Fix $j \in \mathbb{N}$ and fix $x \in \widetilde{\mathcal{L}}(j)$. Then $x \in \mathcal{L}_\rho(0)$ by Lemma 6.1. By hypothesis (A4), $|(u_0)_{xx}(x)| \ge K$. Suppose $(u_0)_{xx}(x) \ge K$. Since $(p_0)_x(x) < \tilde{K}$ by estimate (f) of Lemma 4.2, $(q_0)_x(x) = (u_0)_{xx}(x) - (p_0)_x(x) > 0$. By differentiating (4.7) with respect to x for $j = 1$, and by using the estimates (d), (f) of Lemma 4.2 and (4.12), we get the following estimate:

$$\begin{aligned} q_x^{m,1}(x) - q_x^{m,0}(x) &= \{-[\sigma(u_x^{m,1}(x))]_x + u^{m,1}(x)\}m \\ &= \{-\sigma'(u_x^{m,1}(x))(p_x^{m,1}(x) + q_x^{m,1}(x)) + u^{m,1}(x)\}m \\ &\ge -\sigma'(u_x^{m,1}(x))q_x^{m,1}(x)m - C_6 m \end{aligned}$$

for some $C_6 > 0$. Hence,

$$(1 + \sigma'(u_x^{m,1}(x))m)q_x^{m,1}(x) \ge q_x^{m,0}(x) - C_6 m.$$

Since m is sufficiently small and $q_x^{m,0} = (q_0)_x > 0$, $q_x^{m,1}$ is also positive. Therefore, the inequality

$$(1 - \sigma_0 m)q_x^{m,1}(x) \ge (1 + \sigma'(u_x^{m,1}(x))m)q_x^{m,1}(x) \ge q_x^{m,0}(x) - C_6 m$$

holds. Recall that $\sigma_0 = \min_{[-\rho,\rho]} |\sigma'|$. By induction, suppose $q_x^{m,j-1} > 0$. Then $q_x^{m,j} > 0$ and

$$(1-\sigma_0 m) q_x^{m,j}(x) \geq q_x^{m,j-1}(x) - C_6 m.$$

By iterating this, we obtain

$$\begin{aligned}
q_x^{m,j} &\geq \frac{1}{1-\sigma_0 m} \cdot q_x^{m,j-1} - C_6 m \cdot \frac{1}{1-\sigma_0 m} \\
&\geq \frac{1}{1-\sigma_0 m} \cdot \left[\frac{1}{1-\sigma_0 m} \cdot q_x^{m,j-2} - C_6 m \cdot \frac{1}{1-\sigma_0 m} \right] - C_6 m \cdot \frac{1}{1-\sigma_0 m} \\
&= \frac{1}{(1-\sigma_0 m)^2} \cdot q_x^{m,j-2} - C_6 m \left[\frac{1}{1-\sigma_0 m} + \frac{1}{(1-\sigma_0 m)^2} \right] \\
&\cdots \\
&= \frac{1}{(1-\sigma_0 m)^j} \cdot (q_0)_x - C_6 m \left[\frac{1}{1-\sigma_0 m} + \cdots + \frac{1}{(1-\sigma_0 m)^j} \right] \\
&= \frac{1}{(1-\sigma_0 m)^j} \cdot \left((q_0)_x - \frac{C_6}{\sigma_0} \right) + \frac{C_6}{\sigma_0}.
\end{aligned}$$

This implies

$$q_x^{m,j} \geq e^{jm\sigma_0} \cdot (q_0)_x.$$

Therefore, we can establish the exponential growth of q_x^j, that is,

$$|q_x^j(x,t)| \geq C_0 e^{jm\sigma_0} \cdot |(q_0)_x|$$

for some $C_0 > 0$. Similarly, we get the same conclusion for the case $(u_0)_{xx}(x) \leq -K$, and this proves (i) of Lemma 6.2.

Note that for $K > 4\tilde{K}$,

$$|q_x^j(x,t)| \geq C_0 e^{jm\sigma_0} \cdot |(q_0)_x| \geq C_0 e^{jm\sigma_0} (|(u_0)_{xx}| - \tilde{K}) \geq 3K_0 e^{jm\sigma_0}. \tag{6.1}$$

Here, $K_0 = \tilde{K} C_0$. If $q^j = \rho_0$, then $u_x^j = p^j + q^j = p^j + \rho_0 \geq -\eta + \rho_0 > 0$, and if $q^j = -\rho_0$, then $u_x^j = p^j - \rho_0 \leq \eta - \rho_0 < 0$, which implies $sign(u_x^j) = sign(q^j)$ at $|q^j| = \rho_0$. By using this and (4.7), and also by using estimates (b), (e) of Lemma 4.2, we have the estimate

$$\begin{aligned}
\frac{d}{dt}|q^j(x,t)| &= sign(q^j(x,t)) \cdot \left[\frac{q^{m,j}(x) - q^{m,j-1}(x)}{m} \right] \\
&= sign(u_x^j(x,t)) \cdot \left[\sigma(0) - \sigma(u_x^{m,j}(x)) + \int_0^1 \sigma(u_x^{m,j}) + \pi_a \left(\int_0^x u^{m,j} \right) \right] \\
&\geq -\sigma'(c') \cdot u_x^j \cdot sign(u_x^j) - \sigma'(c') \cdot (u_x^{m,j} - u_x^j) \cdot sign(u_x^j) - 2\sigma_0 \eta \\
&\geq \sigma_0 \cdot |u_x^j| - 2\sigma_0 \eta - \sigma'(c') \cdot sign(u_x^j) \cdot \frac{jm-t}{m} (u_x^{m,j} - u_x^{m,j-1}) \\
&\geq \sigma_0 \cdot (\rho - 4\eta) - \sigma'(c') \cdot sign(u_x^j) \cdot \frac{jm-t}{m} (u_x^{m,j} - u_x^{m,j-1}) \qquad (6.2)
\end{aligned}$$

at $|q^j| = \rho_0$ and for some c' between 0 and $u_x^{m,j}(x)$. Note that $|\frac{jm-t}{m}| < 1$. By estimate (a) of Lemma 4.2, $|p^{m,j} - p^{m,j-1}| \leq 2\eta \ll 1$. By (4.15), $|q^{m,j} - q^{m,j-1}| = m|q_t^j| \leq$

$mM_1 \ll 1$ when $m \ll 1$. Hence, $|u_x^{m,j} - u_x^{m,j-1}| \ll 1$, and this enables the second term of (6.2) to be small. Therefore,

$$\frac{d}{dt}|q^j(x,t)| \geq 0,$$

which implies $|q^j(x,t)| \leq |q^{j+1}(x,t)|$ for all $j \in \mathbb{N}$ when $|q^j(x,t)| = \rho_0$. By (i), q^j is strictly increasing or decreasing on $\widetilde{\mathcal{L}}(j)$, which implies (ii). ◻

From part (i) of Lemma 6.2, estimate (f) of Lemma 4.2, and the hypothesis (A4),

$$\begin{aligned}
|u_{xx}^j(x,t)| &\geq |q_x^j(x,t)| - |p_x^j(x,t)| \\
&\geq C_0 e^{jm\sigma_0} \cdot |(q_0)_x| - \tilde{K} \\
&\geq C_0 e^{jm\sigma_0} \cdot (|(u_0)_{xx}| - 2\tilde{K}) \\
&\geq \frac{1}{2} K_0 e^{jm\sigma_0} \qquad (6.3)
\end{aligned}$$

if $x \in \mathcal{L}_{\frac{\rho}{2}}(j)$ and $K > 4\tilde{K}$.

From (6.3) and the fact that $\|u^{m,j}\|_{C^2} < \infty$ for all $j \in \mathbb{N}$, $\mathcal{L}_{\frac{\rho}{2}}(j)$ has a finite number of components $[a_i^m(j), b_i^m(j)]$, $0 < a_1^m(j) < b_1^m(j) < \cdots < a_N^m(j) < b_N^m(j) < 1$, in each of which $u_x^j(x,t)$ is strictly monotone and has exactly one zero $x_i^m(j)$. Also, $N(j) \geq 1$ since $u^j(0,t) = u^j(1,t) = 0$ for all $j \in \mathbb{N}$.

LEMMA 6.3. $N(j) \equiv$ const. *for all* $j \in \mathbb{N}$.

Proof. For all $j \in \mathbb{N}$, define

$$g^j(x,t) := u_x^j(x,t), \quad (j-1)m < t \leq jm.$$

Since $g^j, g_x^j \in C((0,1) \times ((j-1)m, jm])$ and at each zero (x_0, t_0) of g^j, $|g_x^j(x,t)| \geq \frac{K_0}{2} > 0$ by inequality (6.3), $\{g^j(x_0,t_0) | (x_0,t_0)$ is a zero of $g^j\}$ does not contain a critical value of $g^j(\cdot,t)$ for each t_0. By the implicit function theorem, the number of zeros of $g^j(\cdot,t)$ is independent of t for $(j-1)m < t \leq jm$ for all $j \in \mathbb{N}$. ◻

Similarly, by defining

$$g^j(x,t) := u_x^j(x,t) - \frac{\rho}{2} \quad \text{and} \quad g^j(x,t) := u_x^j(x,t) + \frac{\rho}{2},$$

the number of connected components of $\mathcal{L}_{\frac{\rho}{2}}(j)$ is independent of j. Now, the proof of (P1) and (P2) is complete.

From Lemma 6.1, $[a_i^m(j), b_i^m(j)] \subseteq [(a_0)_i, (b_0)_i]$, $i = 1, \ldots, N$. Moreover,

$$\begin{aligned}
\rho &= |u_x^j(b_i^m(j),t) - u_x^j(a_i^m(j),t)| \\
&= \int_{a_i^m(j)}^{b_i^m(j)} |u_{xx}^j| dx \\
&\geq \frac{1}{2} K_0 e^{jm\sigma_0} \cdot |b_i^m(j) - a_i^m(j)|,
\end{aligned}$$

which implies

$$|b_i^m(j) - a_i^m(j)| \leq \frac{2\rho}{K_0} \cdot e^{-jm\sigma_0} \quad \text{for all } i = 1, \ldots, N$$

for fixed j and $K > 4\tilde{K}$. This proves the last part of (P3). The rest of (P3) was already proved.

From (6.1) and from similar analysis as in the case $\mathcal{L}_{\frac{\rho}{2}}(j)$, $\widetilde{\mathcal{L}}(j)$ has a finite number of components $[\alpha_i^m(j), \beta_i^m(j)],\ 0 < \alpha_1^m(j) < \beta_1^m(j) < \cdots < \alpha_N^m(j) < \beta_N^m(j) < 1$. By Lemma 6.1, $x_i^m(j) \in [a_i^m(j), b_i^m(j)] \subseteq [\alpha_i^m(j), \beta_i^m(j)] \subseteq [a_i^0, b_i^0]$. By (ii) of Lemma 6.2, $[\alpha_i^m(j+1), \beta_i^m(j+1)] \subseteq [\alpha_i^m(j), \beta_i^m(j)]$. Therefore, the set of $[\alpha_i^m(j+1), \beta_i^m(j+1)]$ forms a nested family of intervals. Hence,

$$\begin{aligned} 2\rho > 2\rho_0 &= |q^j(\beta_i^m(j), t) - q^j(\alpha_i^m(j), t)| \\ &= \int_{\alpha_i^m(j)}^{\beta_i^m(j)} |q_x^j| dx \\ &\geq 3K_0 |\beta_i^m(j) - \alpha_i^m(j)| \cdot e^{jm\sigma_0}, \end{aligned}$$

which concludes the proof of (P4).

(P3) and (P4) automatically imply that $(u_\star^m)_x$ is discontinuous at every $(x_\star)_i^m$. It remains now to show that $(u_\star^m)_x$ is continuous on $(0,1)\backslash\{(x_\star)_1^m, \ldots, (x_\star)_N^m\}$. Since $u_\star^m$ is an equilibrium state, it satisfies the equation

$$\sigma((u_\star^m)_x(x)) = \int_0^x (u_\star^m) + \lambda^m$$

for some constant $\lambda^m > 0$. We know that the first term on the right-hand side of the above equation is small by estimate (b) of Lemma 4.2. Furthermore, λ^m is sufficiently small on $(0,1)\backslash\{(x_\star)_1^m, \ldots, (x_\star)_N^m\}$. Therefore, $(u_\star^m)_x$, the inverse image of σ, is continuous on those intervals, which proves (P5). Theorem 3.1 is finally complete.

Remark. The results of transition layer dynamics work for the discretized viscoelastic system without the elastic foundation term u, that is, for the system

$$\frac{1}{m^2}(u - 2u^{m,j-1} + u^{m,j-2}) - (\sigma(u_x))_x - \frac{1}{m}(u_x - u_x^{m,j-1})_x = 0.$$

The proof is similar to the proof for the system with the elastic foundation. Only the minor change of the proof of energy decay (Lemma 4.1), the proof of Lemma 5.3, and the estimate of $q^j(x,t)$ is needed.

7. Asymptotic behavior of the original system. In this section, we answer the following question: How do our results relate to the asymptotic behavior of the original system (1.1)?

We proved in section 5 that $u^{m,j}$ converges strongly in $W_0^{1,p}$ to a steady state $u_\star^m$ as $j \to \infty$ for fixed $m \ll 1$. Therefore, $u_\star^m$ satisfies

$$-(\sigma(u_x))_x + u = 0.$$

We will show next the existence of a weak limit of $u_\star^{m_k}$ in $W_0^{1,p}$ as $m_k \to 0$ for some sequence $m_k \ll 1,\ k \in \mathbb{N}$, in the following theorem.

Theorem 7.1. *There is a sequence $m_k \ll 1,\ k \in \mathbb{N}$, and $m_k \to 0$ as $k \to \infty$ such that the steady state $u_\star^{m_k}$ in Proposition* 5.1 *converges in $W_0^{1,p}$ to a weak limit $u_\star$ as $k \to \infty$.*

Proof. The difficulty arises due to the nonlinearity of σ. However, by the fact that $(u_\star^m)_x$ is uniformly bounded by $\tilde{K} > 0$ and the coercivity condition for σ in (H2), the inequalities

$$\begin{aligned} \int_0^1 \sigma((u_\star^{m_k})_x) \cdot \zeta_x dx &\leq \hat{M} \int_0^1 (|(u_\star^{m_k})_x|^{p-1} + 1) \cdot \zeta_x dx \\ &\leq \hat{M} \int_0^1 (\tilde{K}^{p-1} + 1) \cdot \zeta_x dx \end{aligned}$$

hold for some $\hat{M} > 0$ and for any test function $\zeta \in C_0^\infty((0,1),\mathbb{R})$. The result follows from the dominated convergence theorem. □

Note that we can assume that σ is globally Lipschitz continuous since $u_x^{m,j}$ is uniformly bounded for all $j \in \mathbb{N}$ and for any $m \ll 1$. Then by [15, section 5.1], the weak solution of the system (1.1) is unique. Combining this with the results shown in [15, Theorem 4.1] and [16, Theorem 3.1] , the discretized solution $u^{m,j}$ converges in $W_0^{1,p}$ to a unique weak solution u of (1.1) as $m \to 0$, and u converges strongly in $W_0^{1,p}$ to a unique equilibrium state u_∞ as $t \to \infty$.

If the weak limit $u_\star$ is unique and is the same as u_∞, the same asymptotic behavior will hold for the system (1.1). However, we do not know the answer to this question and it remains as an open problem.

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